

A Thesis Submitted for the Degree of PhD at the University of Warwick

Permanent WRAP URL:

<http://wrap.warwick.ac.uk/96045>

Copyright and reuse:

This thesis is made available online and is protected by original copyright.

Please scroll down to view the document itself.

Please refer to the repository record for this item for information to help you to cite it.

Our policy information is available from the repository home page.

For more information, please contact the WRAP Team at: wrap@warwick.ac.uk

STABILITY PROPERTIES OF ONE PARAMETER

FLOWS

by

R. F. Thomas

Thesis submitted for Ph.D. degree at Warwick
University.

June, 1979.

ABSTRACT

This thesis is in two parts. The first part consists of chapters 1-4 and the second part is chapter 5. In the first part we consider the idea of approximating pieces of orbits by a single orbit. There are many examples of such properties (*approximation property* (A.P.) see chapter one, definitions 1.2, 1.4), (*Specification property* (S.P.), [24]), (*pseudo orbit tracing property* (P.O.T.P.) [25] and definition 4.1).

In chapter one, we show that (A.P.) for a homeomorphism (flow) is equivalent to topological transitivity and density of periodic points and how this property (A.P.) is invariant under topological conjugacy. In theorem 1 we prove that an expansive homeomorphism which is topologically mixing and has P.O.T.P. also has the S.P.

In chapter two, theorem 2, we prove that the P.O.T.P. for flows is invariant under topological conjugacy with preserved orientation (velocity changes). Also we prove in theorem 3 that the suspension flow [2] for a homeomorphism $T:X \rightarrow X$ has P.O.T.P. if and only if T has the P.O.T.P.

In chapter three, we prove that an expansive flow which has the P.O.T.P. is topologically stable (Theorem 4).

In chapter four, the last theorem in this part is that every flow without fixed points on a compact manifold M which is topologically stable has the P.O.T.P. Then some important corollaries are deduced.

In the second part of this thesis, we define an I-hyperbolic toral automorphism $A:T^n \rightarrow T^n$ to be a hyperbolic toral automorphism whose matrix $[A]$ of integers has irreducible characteristic polynomial over the field of rational numbers. A codimension 1 hyperbolic toral automorphism is an example of an I-hyperbolic toral automorphism. Let W_A^S (W_A^u) denote the stable (unstable) manifold at 0 and $N(A)$ set of normal points with respect to A . We prove for any two codimension 1 hyperbolic toral automorphisms A, B of T^n that

- (a) $N(A) = N(B) \implies W_A^S = W_B^S, W_A^u = W_B^u$ and $AB = BA$;
- (b) $AB = BA \implies AN(B) = N(B) \implies AW_B^S = W_B^S \implies AB = BA$.

Finally, some interesting applications are given.

ACKNOWLEDGEMENTS

I would like to thank my supervisor Dr. Peter Walters for his help, encouragement and understanding. I would also like to thank Professor William Parry for being very helpful during the absence of my supervisor.

Finally, I would like to thank the "Ministry of Higher Education of Iraq", for the financial support throughout the years of my studies.

TABLE OF CONTENTS

<u>INTRODUCTION</u>	i
<u>CHAPTER I</u>	1
1.1. Introduction	1
1.2. Topological Transitive and Approximation Property	1
1.3. Change of Velocity and Suspension Flows	7
1.4. Specification Property	10
1.5. Pseudo Orbit Tracing Property	15
<u>CHAPTER II</u>	21
2.1. Pseudo Orbit Tracing Property	21
2.2. P.O.T.P. and Topological Conjugacy	31
2.3. The Suspension Flow	35
<u>CHAPTER III</u>	43
<u>CHAPTER IV</u>	68
<u>REFERENCES</u>	85
<u>CHAPTER V</u>	88
5.1. Introduction	88
5.2. Notation and Definitions	90
5.3. Anosov Diffeomorphisms and Stable (unstable) Manifolds	93
5.4. Measures and Weak Topology	99
5.5. Uniform Distribution Modulo 1	100
5.6. Normal Points	104
5.7. Applications	112
5.8. References	117

INTRODUCTION

The idea of approximating pieces of orbits by a single orbit with a certain uniformity is one of the most important dynamical properties of axiom A maps and of hyperbolic flows (Anosov flows). There are many examples of such properties which have recently been studied in ergodic theory and statistical mechanics, the first and simplest idea of such properties is the *approximation property* (see chapter one, definitions 1.2, 1.4) and we show how this property for a homeomorphism (flow) is equivalent to the definition of topological transitivity *and density of periodic points*. Since the topological transitivity property invariant under topological conjugacy (in particular velocity changes) for a flow ϕ , so is the approximation property. Moreover, after giving definition 1.6 (suspension flow of a homeomorphism) it is clear that the suspension flow of a homeomorphism $T:X \rightarrow X$ under a continuous map $f:X \rightarrow \mathbb{R} > 0$ has the approximation property. In fact the suspension flows are important examples of flows whose behaviour resemble that of homeomorphisms. The second example of such a property, which is stronger than the above property, is the *specification property* (definitions 1.7, 1.9) [24], [3], [4]. This property has been used in [24], [4], [21],[22] and [23]. In proposition 1.3 we show that the suspension flow of a homeomorphism $T:X \rightarrow X$ under any *constant* map $f:X \rightarrow \mathbb{R} > 0$ never satisfies the specification

property even if T has the specification property. The third property, which is the most important, is the *tracing of pseudo orbits* for a homeomorphism; that is, after applying a homeomorphism T to the point x_i with a small perturbation to x_{i+1} one allows an orbit say $\{T^i x\}$ to give a perturbation for this sequence $\{x_i\}$. As we know necessary conditions for the specification property are topologically mixing and density of periodic points ([24], proposition 2), so one can say that the tracing of pseudo orbits is in some sense weaker property than the specification property. Many important results can be deduced from the pseudo orbit tracing property. P. Walters [25] shows that a subshift has this property (P.O.T.P.) if and only if it is of finite type; that this property together with expansiveness [1] implies topological stability; and ~~that~~ the topologically stable homeomorphisms on a manifold of dimension ≥ 2 have the pseudo orbit tracing property. Also in this thesis we proved in chapter one theorem 1, that an expansive homeomorphism which is topologically mixing has pseudo orbit tracing property also has the specification property. Therefore the topological entropy of such a homeomorphism is positive.

Smale's axiom A flows are those where the nonwandering set Ω is hyperbolic and Ω is the closure of the set of periodic orbits. For studying the topological stability of such flows in this work, the technical idea needed is the approximating of pieces of orbits by a single orbit. This is called the *pseudo orbit tracing property* for flows, and the idea to approximate

(iii)

pieces of orbits each of time length ≥ 1 (i.e. $(\phi_t x_i)_{0 \leq t \leq t_i}, t_i \geq 1$) with the extra property $\phi_{t_i} x_i$ is close to x_{i+1} by an orbit say $(\phi_t z)_{t \in \mathbb{R}}$, so that $\phi_t z$ is close to $\phi_t x_i$ for $0 \leq t \leq t_i$ (definitions 1.10, 2.1). This important technique has been used in [10], [12], [13] and [14]. In this thesis we are going to show the pseudo orbit tracing property and topological stability for a flows are related.

In chapter two, the definition of pseudo orbit tracing property with respect to time $a > 0$ (i.e., the chain $(\{x_i\}_{-\infty}^{\infty}, \{t_i\}_{-\infty}^{\infty})$ has the property $t_i \geq a$ for all $i \in \mathbb{Z}$) is given (definitions 2.1, 2.2) and we show in proposition 2.2 that a flow ϕ has the pseudo orbit tracing property with respect to time $a > 0$ if and only if has pseudo orbit tracing property with respect to time 1. In theorem 2, we prove that this property (pseudo orbit tracing property) is invariant under topological conjugacy with preserved orientation, in particular velocity changes. Also we prove in theorem 3 that the suspension flow for a homeomorphism $T: X \rightarrow X$ under a continuous map $f: X \rightarrow \mathbb{R} > 0$ has the pseudo orbit tracing property if and only if T has the pseudo orbit tracing property. This is one reason why the pseudo orbit tracing property for flows is important to study.

In chapter three, another property is needed (expansiveness [2]) which is also a property of hyperbolic flows and we prove in theorem 4 that an expansive flow which has the pseudo orbit

tracing property is topologically stable. Moreover in theorem 5 we show that if the perturbation flow in the statement of theorem 4 is also expansive, then the conjugating map which makes the given flow in theorem 4 topologically stable is injective.

In chapter four, we study when topological stability for a flow implies the pseudo orbit tracing property. For this we need to define a *finite pseudo orbit tracing property* for flows (definition 4.1). This idea is important because it is easier to check that a flow has the finite pseudo orbit tracing property than the (infinite) pseudo orbit tracing property. So because of this we prove in proposition 4.1 that every flow without fixed points having the finite pseudo orbit tracing property also has the pseudo orbit tracing property (i.e. infinite pseudo orbit tracing property). The last important theorem in this chapter is that every flow without fixed points on a compact manifold M which is topologically stable has the pseudo orbit tracing property. Using this work particularly chapters 2, 3 and 4 one can show that the topological stability property for flows is invariant under topological conjugacy with preserved orientation, in particular velocity changes. Finally, P. Walters' work [25] together with this thesis, shows that if ϕ is the suspension flow of a homeomorphism T on a compact manifold M under a continuous map $f:M \rightarrow \mathbb{R} > 0$, then

- (i) T is topologically stable if ϕ is topologically stable.
- (ii) ϕ is topologically stable if dimension of $M \geq 2$ and T is topologically stable.

The references for chapters 1 - 4 occur after chapter 4.

CHAPTER I

1.1. INTRODUCTION

The most important dynamical property is to approximate simultaneously a family of pieces of orbits by an orbit. Since there are many ideas of such approximations we are going to show how these approximations are related to each other.

1.2. TOPOLOGICALLY TRANSITIVE AND APPROXIMATION PROPERTY

X will always denote a compact metric space with metric d and $T:X \rightarrow X$ will always be a homeomorphism unless otherwise stated. A point $x \in X$ is said to be a periodic point of T if $T^n x = x$ for some $n \neq 0$. The least such positive n with this property will be called the period of x under T .

Definition 1.1.

$T:X \rightarrow X$ is topologically transitive if there exists a point $x \in X$ such that the orbit of x $\{T^n x\}_{n=-\infty}^{\infty}$ is dense in X . [26],[27].

Theorem (cf [26] Theorem (5.4))

The following are equivalent:

1. T is topologically transitive.
2. $TE = E$, E is closed, $E \neq X \implies E$ is nowhere dense.
3. If U, V are non-empty open sets then there exists an integer n such that $T^n U \cap V \neq \emptyset$.

4. $\{x \in X: \text{closure of the orbit of } x \text{ is not } X\}$ is a set of first category.

By a string $A = [a, b]$ we mean a finite set of consecutive integers $\{a, a + 1, \dots, b\}$. By a piece of orbit we mean a set $\{T^i x: x \in X, i \in A\}$.

Definition 1.2.

(X, T) has the approximation property (A.P.) if for every $\epsilon > 0$ and any integer $k > 1$ and for any sequence of points $\{x_i\}_0^k$ in X and for any set of strings $A_i = [a_i, b_i]$, $i = 0, 1, \dots, k$ with $a_{i+1} - b_i \geq 1$, $i = 0, 1, \dots, k-1$, there is a periodic point $z \in X$ of period $p > b_k - a_0$ and a sequence of integers $\{d_i\}_0^k$ with $d_0 = 0$ such that $d(T^{j+d_i} z, T^j x_i) < \epsilon$ for $j \in A_i$, $i = 0, 1, 2, \dots, k$. (This definition does not depend on the choice of metric d .)

Proposition 1.1.

(X, T) satisfies the approximation property if and only if the periodic points are dense and T is topologically transitive.

Proof: Assume (X, T) has A.P., obvious that the periodic points are dense. Let V and U be any two non-empty open sets such that $x_0 \in V$ and $x_1 \in U$ and take ϵ small enough such that the neighbourhood of x_0 of radius ϵ is contained in V and the neighbourhood of x_1 of radius ϵ is contained in U , consider the strings $A_0 = \{0\}$, $A_1 = \{1\}$ with two points $x_0, T^{-1}x_1$. Using the hypothesis there

is a point z and an integer w such that $d(z, x_0) < \varepsilon$ and $d(T^{1+w}z, x_1) < \varepsilon$. So $z \in V$ and $T^{1+w}z \in U$ implies $T^{-1-w}U \cap V \neq \emptyset$. Conversely, assume the periodic points are dense in X and T is topologically transitive. Given $\varepsilon > 0$ and an integer $k > 1$ and a set of strings $A_i = [a_i, b_i]$, $i = 0, 1, \dots, k$ with $a_{i+1} - b_i \geq 1$, $i = 0, 1, \dots, k-1$. Let $\varepsilon' < \varepsilon$ have the property that $d(x, y) < \varepsilon'$ implies $d(T^i x, T^i y) < \varepsilon/2$ for $i = a_0, a_0 + 1, \dots, b_k$ and for every $x, y \in X$ and let $z' \in X$ be such that orbit of z' is dense in X . Without loss of generality assume $d(z', x_0) < \varepsilon'$, it is clear that there is a sequence of positive integers $\{d_i\}_1^k$ such that $d(T^{d_i} z', x_i) < \varepsilon'$, $i = 1, 2, \dots, k$. Since the periodic points are dense, choose z to be a periodic point near z' such that $d(T^i z, T^i z') < \varepsilon/2$ for $i = a_0, a_1 + 1, \dots, b_k + \sum_{j=1}^k d_j$. Hence by taking $d_0 = 0$ we have

$$d(T^{j+d_i} z, T^j x_i) < \varepsilon \text{ for } j \in A_i. \quad \square$$

Let (X, d) be a compact metric space and $\phi: X \times \mathbb{R} \rightarrow X$ be a continuous flow on X (i.e., ϕ is continuous and $\phi(x, t+s) = \phi(\phi(x, t), s)$). Let ϕ_t denote the homeomorphism of X defined by $\phi_t(x) = \phi(x, t)$. A point $x \in X$ is said to be a periodic point for ϕ if $\phi_t x = x$ for some real number $t \neq 0$. The least such positive t with this property will be called the period of x under ϕ . If $\phi_t x = x$ for every real number t , then x will be called a fixed point under ϕ .

Definition 1.3.

A flow $\phi: X \times \mathbb{R} \rightarrow X$ is topologically transitive if there exists a point $x \in X$ such that the orbit of x $((\phi_t x)_{t \in \mathbb{R}})$ is dense in X . [11]

It is well known that ϕ is topologically transitive if and only if for every non-empty open sets U, V and for every $r > 0$, there exists $t \geq r$ such that $\phi_{-t} U \cap V \neq \emptyset$. [11]

Definition 1.4.

A flow ϕ on a compact metric space X has approximation property (A.P.) if for every $\epsilon > 0$ and any integer $n > 1$ and for every pair of sequences $\{r_i\}_0^{n-1}, \{t_i\}_0^n$ with $r_i > 0, t_i \geq 0$ of real numbers and for every sequence $\{x_i\}_0^n$ of points in X , there exists a periodic point $z \in X$ of period $p > \sum_0^n t_i + \sum_0^{n-1} r_i$ and a continuous map α from a closed interval $[0, P]$ onto itself with $\alpha(0) = 0, \alpha(P) = P$ such that

$$d[\phi_{\alpha(t)} z, \phi_t x_K] < \epsilon \text{ whenever } \sum_0^{k-1} t_i + \sum_0^{k-1} r_i \leq t \leq \sum_0^k t_i + \sum_0^{k-1} r_i, \\ 0 \leq K \leq n.$$

In this definition we assumed that $\sum_{i=a}^b ()_i = 0$ for $b < a$ and this will be a standard notation throughout this work.

Proposition 1.2.

Let ϕ be a flow on a compact metric space. Then ϕ satisfies the approximation property if and only if the periodic points are dense and ϕ is topologically transitive.

Proof: Assume ϕ has A.P., it is clear that the periodic points are dense. Now let U, V be two non-empty open sets and $r > 0$. Let x_0, x_1 be two points in V, U respectively, take $\epsilon > 0$ such that $d(y, x_0) < \epsilon$ implies $y \in V$, $d(y, x_1) < \epsilon$ implies $y \in U$. Now, let $r > 0$ be a real number, $t_0 = t_1 = 0$, take the two points $x_0, \phi_{-r}x_1$. Then there exists a periodic point z of period $p > r$ and a continuous map $\alpha: [0, p] \rightarrow [0, p]$ with $\alpha(0) = 0, \alpha(p) = p$ such that $d(z, x_0) < \epsilon, d(\phi_{\alpha(r)}z, x_1) < \epsilon$, where $[0, p]$ is a closed interval in \mathbb{R} . Hence $z \in V$ and $z \in \phi_{-\alpha(r)}U$ implies $\phi_{-\alpha(r)}U \cap V \neq \emptyset$. Conversely, assume the periodic points are dense and ϕ topologically transitive. Take $\epsilon > 0, n > 1$ integer, and sequences of real numbers $\{r_i\}_0^{n-1}, \{t_i\}_0^n$ with $r_i > 0, t_i \geq 0$, and $\{x_i\}_0^n$ sequence of points in X . Choose $\epsilon' < \epsilon$ such that $d(x, y) < \epsilon'$ implies $d(\phi_u x, \phi_u y) < \epsilon/2$ for all $0 \leq u \leq \sum_0^n t_i + \sum_0^{n-1} r_i$. Since ϕ is topologically transitive, there exists $z \in Y$ such that the orbit $(\phi_t z)_{t \in \mathbb{R}}$ is dense in X ; there are $\lambda_0, \lambda_1, \dots, \lambda_n$ real numbers (without loss of generality take $\lambda_0 = 0$) such that $d(\phi_{\lambda_i} z, x_i) < \epsilon'$ for $0 \leq i \leq n$, therefore $d[\phi_{u+\lambda_i} z, \phi_u x_i] < \epsilon/2$ for $\sum_0^{i-1} t_j + \sum_0^{i-1} r_j \leq u \leq \sum_0^i t_j + \sum_0^{i-1} r_j, 0 \leq i \leq n$. Since the periodic points are dense, if we take $\epsilon'' < \epsilon$ such that $d(x, y) < \epsilon''$ implies

$d(\phi_{u+\lambda_i} x, \phi_{u+\lambda_i} y) < \varepsilon/2$ for $0 \leq u \leq \sum_{j=0}^n t_j + \sum_{j=0}^{n-1} r_j$. Choose a periodic point z' with period $p > \sum_{j=0}^n t_j + \sum_{j=0}^{n-1} r_j$ such that

$d(z', z) < \varepsilon''$, therefore

$$d[\phi_{u+\lambda_i} z', \phi_{u+\lambda_i} z] < \varepsilon/2 \text{ for } \sum_{j=0}^{i-1} t_j + \sum_{j=0}^{i-1} r_j \leq u \leq \sum_{j=0}^i t_j + \sum_{j=0}^{i-1} r_j, \text{ all } i.$$

Hence

$$d[\phi_{u+\lambda_i} z', \phi_u x_i] < \varepsilon \text{ for } \sum_{j=0}^{i-1} t_j + \sum_{j=0}^{i-1} r_j \leq u \leq \sum_{j=0}^i t_j + \sum_{j=0}^{i-1} r_j, \text{ all } i.$$

Now define $\alpha: [0, P] \rightarrow [0, P]$ as follows:

$$\alpha(u) = u + \lambda_i \text{ for } \sum_{j=0}^{i-1} t_j + \sum_{j=0}^{i-1} r_j \leq u \leq \sum_{j=0}^i t_j + \sum_{j=0}^{i-1} r_j, 0 \leq i \leq n.$$

$$\text{and } \alpha(u) = \frac{r_i + \lambda_{i+1} - \lambda_i}{r_i} (u - \sum_{j=0}^i t_j - \sum_{j=0}^{i-1} r_j) + \sum_{j=0}^i t_j + \sum_{j=0}^{i-1} r_j + \lambda_i \text{ for}$$

$$\sum_{j=0}^i t_j + \sum_{j=0}^{i-1} r_j \leq u \leq \sum_{j=0}^i t_j + \sum_{j=0}^i r_j, 0 \leq i \leq n-1$$

$$\text{and } \alpha(u) = \frac{\sum_{j=0}^n t_j + \sum_{j=0}^{n-1} r_j + \lambda_n - p}{\sum_{j=0}^n t_j + \sum_{j=0}^{n-1} r_j - p} (u - p) + p \text{ for } \sum_{j=0}^n t_j + \sum_{j=0}^{n-1} r_j \leq u \leq p.$$

Clearly α is continuous, $\alpha(0) = 0$, $\alpha(P) = P$ and for

$$\sum_{j=0}^{i-1} t_j + \sum_{j=0}^{i-1} r_j \leq u \leq \sum_{j=0}^i t_j + \sum_{j=0}^{i-1} r_j, 0 \leq i \leq n \text{ we have}$$

$$d[\phi_{\alpha(u)} z', \phi_u x_i] = d[\phi_{u+\lambda_i} z', \phi_u x_i] < \varepsilon. \quad \square$$

1.3. CHANGE OF VELOCITY AND SUSPENSION FLOWS.

Definition 1.5.

We recall that the two flows ϕ on X and ψ on Y are said to be conjugate if there is a homeomorphism $\lambda: X \rightarrow Y$ mapping orbits of ϕ onto orbits of ψ .

Remark.

Assume the flows ϕ on X and ψ on Y are conjugate. $\lambda: X \rightarrow Y$ is the conjugating homeomorphism. Now fix a point $x \in X$ and take $\lambda(\phi_t x)_{t \in \mathbb{R}}$ which is an orbit of ψ . If x is not periodic under ϕ , the map $\sigma_x: \mathbb{R} \rightarrow \mathbb{R}$ defined by $\lambda \phi_t x = \psi_{\sigma_x(t)} \lambda x$ is a well defined bijection map with $\sigma_x(0) = 0$. σ_x is either strictly increasing or decreasing. Hence σ_x is a homeomorphism of \mathbb{R} . If x is periodic. Let v be the smallest positive real number such that $\phi_v x = x$ and also u be the smallest positive number such that $\psi_u \lambda x = \lambda x$. It is clear that $\sigma_x: [0, v] \rightarrow [0, u]$ or $[-u, 0]$ is a well defined homeomorphism. Similarly σ_x can be defined on $[nv, (n+1)v] \rightarrow [nu, (n+1)u]$ or $[-(n+1)u, -nu]$ and so σ_x becomes a homeomorphism of \mathbb{R} with $\sigma_x(0) = 0$. We call such a map $\sigma: X \times \mathbb{R} \rightarrow \mathbb{R}$ the cocycle of the flow ϕ with values in \mathbb{R} . [19]. The two flows ϕ and ψ in definition (1.5) are said to be conjugate with preserved orientation if they are conjugate and σ_x is strictly increasing for every $x \in X$. An example of such a conjugacy is the changing of velocity (i.e., ψ is obtained from a flow ϕ by changing velocity (see [19], [15])).

Let (Y, d) be a compact metric space and $T: Y \rightarrow Y$ a homeomorphism. Let $f: Y \rightarrow \mathbb{R} > 0$ be continuous.

Definition 1.6.

The suspension of T under f is the flow ϕ_f on the space

$$Y_f = \bigcup_{0 \leq t \leq f(y)} \{(y, t) \mid (y, f(y)) \sim (Ty, 0)\}$$

defined for small non-negative time by $\phi_t(y, s) = (y, t+s), 0 \leq t+s < f(y)$.

Each suspension of T is conjugate to the suspension of ϕ under 1, the constant function with value 1. A homeomorphism from Y_1 to Y_f that conjugates the flows is given by $(y, t) \mapsto (y, tf(y))$. In fact this is also a good example of conjugacy with preserved orientation. For this reason we shall concentrate on suspension under the function 1. We shall now define a metric on Y_1 . (see [2]). Suppose that the diameter of Y under d is less than 1.

Consider the subset $Y \times \{t\}$ of $Y \times [0, 1]$ and let d_t denote the metric on $Y \times \{t\}$ defined by $d_t((y, t), (z, t)) = (1-t)d(y, z) + td(Ty, Tz)$, $y, z \in Y$. Note that $d_0((y, 0), (z, 0)) = d(y, z)$ and $d_1((y, 1), (z, 1)) = d(Ty, Tz)$. Now let $x_1, x_2 \in Y_1$. Consider all finite chains $x_1 = w_0, w_1, w_2, \dots, w_n = x_2$ between x_1 and x_2 where for each i either w_i and w_{i+1} belong to $Y \times \{t\}$ for some t (in such case $[w_i, w_{i+1}]$ is called a horizontal segment) or w_i and w_{i+1} are on the same orbit (and then $[w_i, w_{i+1}]$ is called a vertical segment). Define the length of the chain to be the sum

of the lengths of its segments where the length of a horizontal segment $[w_i, w_{i+1}]$ is measured in the metric d_t if w_i and w_{i+1} belongs to $Y \times \{t\}$, and the length of a vertical segment $[w_i, w_{i+1}]$ is the shortest distance between w_i, w_{i+1} along the orbit (ignoring the direction of the orbit) using the usual metric on R . If $w_i \neq w_{i+1}$ and w_i, w_{i+1} are on the same orbit and on the same set $Y \times \{t\}$ then the length of the segment $[w_i, w_{i+1}]$ is taken to be $d_t(w_i, w_{i+1})$, since this is always less than 1. Then define $\rho(x_1, x_2)$ to be the infimum of the lengths of all chains between x_1 and x_2 . Clearly ρ is a metric on Y_1 . Also ρ gives the topology on Y_1 .

From propositions (1.1), (1.2) we have the following.

Corollary 1.

Assume the flows ϕ on X and ψ on Y are conjugate. Then ϕ has A.P. if and only if ψ has A.P. (if a flow ϕ obtained from a flow ψ by changing velocity and ψ has A.P., so does ϕ).

Corollary 2.

Let ϕ_f on Y_f be the suspension flow of $T:Y \rightarrow Y$ under a continuous map $f:Y \rightarrow R > 0$. Then T has A.P. if and only if ϕ_f has A.P.

1.4. SPECIFICATION PROPERTY

Let X be a compact metric space and $T:X \rightarrow X$ be a homomorphism (i.e., (X,T) is a dynamical system).

Definition 1.7.

(X,T) is said to have the specification property (S.P.) if for any $\epsilon > 0$ and any integer $k > 1$, there is a positive integer $M = M_{\epsilon,k}$ such that for any sequence of points $\{x_i\}_{i=0}^k$ in X , and any set of strings $A_i = [a_i, b_i]$ with $a_{i+1} - b_i > M$, for $i = 0, 1, \dots, k-1$ and any integer $P > b_k - a_1 + M$, there is a periodic point $z \in X$ with period p such that $d(T^j z, T^j x_i) < \epsilon$ for $j \in A_i$, $i = 0, 1, \dots, k$. (This definition does not depend on the choice of the metric d .)

A specification property has been first considered by Bowen in [3], and was used later in [4], [21], [22], [23] and [20]. This seems a very strong condition but there are many examples of dynamical systems satisfying it. (For such examples see [24].)

In [20] Ruelle defines the dynamical system (X,T) to have the strong specification property if M in definition (1.7) is independent of the choice of k . In fact the strong specification property is Bowen's definition in [3], [4]. Also [20] has a weak specification: it is the same as Bowen's except that z is not required to be periodic in definition (1.7) (hence no condition on p). Definition (1.7) is between strong and weak specification.

Proposition. (cf. [24] proposition (3))

If (X,T) is a non-trivial system with the specification property, then the topological entropy of (X,T) is larger than 0. (Also see [4], [26], [5] and [6]). In [24] Sigmund has shown the following.

Lemma 1.1.

(X,T) has S.P. if and only if for every $\epsilon > 0$ there is a positive integer $M = M_\epsilon$ such that for any choice of points $x_0, x_1 \in X$ and strings $A_0 = [a_0, b_0]$, $A_1 = [a_1, b_1]$ with $a_1 - b_0 > M$, and any integer $p > b_1 - a_0 + M$, there exists a periodic point $z \in X$ with period p such that

$$d(T^j z, T^j x_0) < \epsilon \quad \text{for } j \in A_0,$$

$$d(T^j z, T^j x_1) < \epsilon \quad \text{for } j \in A_1.$$

Definition 1.8.

Let $T:X \rightarrow X$ be a homeomorphism, X is compact metric space. (X,T) is topologically mixing if for any pair of non-empty open sets U, V in X there exists an integer $N > 0$ such that $T^{-n}U \cap V \neq \emptyset$ for all $n \geq N$.

Proposition. (cf. [24] proposition (2)).

If (X, T) satisfies the specification property, then

- (a) the periodic points are dense;
- (b) (X, T) is topologically mixing.

The notion of S.P. can be defined also for a flow ϕ on a compact metric space X . (Bowen's definition in [8]).

Definition 1.9.

The flow ϕ on X is said to have the specification property (S.P.) if for any $\epsilon > 0$ and $k \geq 1$, there is a positive integer $M = M_{\epsilon, k}$ such that for any $z_0, z_1, \dots, z_k \in X$ and $t_0, t_1, \dots, t_{k+1} \in \mathbb{R}$ with $t_{i+1} \geq t_i + M$, there is an $x \in CO_{\epsilon}^*(t_{k+1} - t_0)$ with

$$d(\phi_{t_i+u}(x), \phi_u(z_i)) < \epsilon \text{ for } 0 \leq u \leq t_{i+1} - t_i - M, 0 \leq i \leq k.$$

Where CO_{ϵ}^* is the set of all periodic points of ϕ with period in the interval $[t-\epsilon, t+\epsilon]$ (see [4]).

Note:

If $T: Y \rightarrow Y$ is a homeomorphism and has S.P., then its suspension flow ϕ_f on Y_f under a ~~constant map~~ $f: Y \rightarrow \mathbb{R} > 0$ never has S.P., to show this fact we need:

Lemma 1.2.

If $\phi: Y \times \mathbb{R} \rightarrow Y$ is the suspension flow of $T: X \rightarrow X$ under a constant map 1 (i.e., $X_1 = Y$ and $\phi_1 = \phi$) and if

$\rho((x,s),(y,t)) < \varepsilon < \frac{1}{4}$, then either $|s-t| < \varepsilon$ or $|1+s-t| < \varepsilon$ or $|1+t-s| < \varepsilon$.

Moreover (a) $t = 0$ implies either $|s| < \varepsilon$ or $|1-s| < \varepsilon$,

(b) $t = \frac{1}{2}$ implies $|s - \frac{1}{2}| < \varepsilon$.

Proof: Assume d is the metric on X and ρ is the metric on Y taken by definition and assume $|s-t| \geq \varepsilon$, $|1+s-t| \geq \varepsilon$ and $|1+t-s| \geq \varepsilon$. If X has fixed point, that is $TZ = Z$, then

$$\rho((x,s),(y,t)) \geq \rho(z,s),(z,t)) \geq \varepsilon.$$

This is a contradiction. If X has no fixed point, then take $X' = X \cup \{\lambda\}$ disjoint union (or i.e., λ is a point not in X) and define d' be a metric on X' such that $d'|_{X \times X} = d$, $d(x,\lambda) = 1$ for all $x \in X$ (i.e., λ is an isolated point in this topology). Define $T':X' \rightarrow X'$ such that $T'/X = T$, $T'(\lambda) = \lambda$ and let $\phi':Y' \times \mathbb{R} \rightarrow Y'$ be the suspension of $T':X' \rightarrow X'$ under a constant map 1. If ρ' is the metric on Y' taken by definition 1.6, then we have

(a) $Y' = Y \cup (\phi'_t(z,0))_{t \in \mathbb{R}}$, where $(\phi'_t(z,0))_{t \in \mathbb{R}}$ is the only periodic orbit with period 1 for ϕ' .

(b) $\phi'/Y \times \mathbb{R} = \phi$

(c) $\rho'/Y = \rho$

Hence for every $x,y \in X$ we have

$$\rho((x,s),(y,t)) = \rho'((x,s),(y,t)) \geq \rho'((z,s),(z,t)) \geq \varepsilon$$

and also this is a contradiction. \square

Proposition 1.3.

If $\phi: Y \times \mathbb{R} \rightarrow Y$ is the suspension flow of a homeomorphism $T: X \rightarrow X$ under a constant map 1, then ϕ does not satisfy the specification property.

Proof: Assume ϕ satisfies definition 1.9 (S.P.). Given

$0 < \epsilon < \frac{1}{4}$ and $k = 2$, using definition (1.9) there is $M > 0$ such that if we take the two points $(x, 0), (x, \frac{1}{2}) \in Y$ and t_0, t_1, t_2 are integers with $t_1 \geq t_0 + M$, $t_2 \geq t_1 + M$, there is $(z, s) \in Y$ periodic such that

$$(a) \quad \rho(\phi_{u+t_0}(z, s), \phi_u(x, 0)) < \epsilon \quad \text{for } 0 \leq u \leq t_1 - t_0 - M,$$

$$(b) \quad \rho(\phi_{u+t_1}(z, s), \phi_u(x, \frac{1}{2})) < \epsilon \quad \text{for } 0 \leq u \leq t_2 - t_1 - M.$$

From (a) and lemma (1.2) if we let $u = 0$ we will have either $|s| < \epsilon < \frac{1}{4}$ or $|1-s| < \epsilon < \frac{1}{4}$. From (b) and lemma (1.2) if we let $u = 0$ we will have $|s - \frac{1}{2}| < \epsilon < \frac{1}{4}$, and this is a contradiction. \square

From this proposition it was clear that even if a homeomorphism $T: X \rightarrow X$ has S.P., then its suspension flow under any constant map $f: X \rightarrow \mathbb{R} > 0$ does not satisfy S.P..

1.5. PSEUDO ORBIT TRACING PROPERTY

Rufus Bowen has stated that the tracing of pseudo orbits is the most important dynamical property of Axiom A map. [7], [8].

X will always denote a compact metric space with metric d and $T:X \rightarrow X$ be a homeomorphism. Let $\delta > 0$. A sequence $\{x_n\}_{n=-\infty}^{\infty}$ in X is said to be a " δ -pseudo orbit" for $T:X \rightarrow X$ if $d(Tx_n, x_{n+1}) < \delta$ for every integer n . For T to be "stable" we would like each pseudo-orbit for T to be closely related to an actual orbit for T . So a δ -pseudo orbit $\{x_n\}_{n=-\infty}^{\infty}$ is said to be ϵ -traced by $x \in X$ if $d(T^n x, x_n) < \epsilon$ for every integer n .

Definition 1.10. [9], [8], [25].

T is said to have the "pseudo orbit tracing property" (P.O.T.P.) if for any $\epsilon > 0$ there is $\delta > 0$ such that each δ -pseudo orbit for T is ϵ -traced by some point of X .

This definition does not depend on the choice of metric on X , and it is preserved under topological conjugacy. To get a feeling for this definition see Walters [25].

Definition 1.11. [10].

A homeomorphism $T:X \rightarrow X$ is expansive if there exists $e(T) > 0$ such that if $d(T^n x, T^n y) \leq e(T)$ for every integer $n \in \mathbb{Z}$, then $x = y$. Such numbers $e(T)$ are called expansive constants.

Definition 1.12. [7], [25]

A homeomorphism $T:X \rightarrow X$ is topologically stable if for every $\epsilon > 0$ there exists $\delta > 0$ so that if $S:X \rightarrow X$ is a homeomorphism with $d(S,T) < \delta$ then there is a continuous map $h:X \rightarrow X$ with $hS = Th$ and $d(h, id) < \epsilon$. Here $d(S,T)$ denotes $\sup_{x \in X} d(S(x), T(x))$.

In [25] Walters proved the following important results.

Theorem. (cf [25] theorem (4)).

An expansive homeomorphism $T:X \rightarrow X$ with the P.O.T.P. is topologically stable.

Theorem. (cf [25] theorem (11)).

Let $T:M \rightarrow M$ be a topologically stable homeomorphism of a compact manifold of dimension ≥ 2 . Then T has the P.O.T.P.

We are going to close this chapter by showing the following.

Theorem 1.

If a homeomorphism $T:X \rightarrow X$ is expansive and has P.O.T.P. and is topologically mixing, then T has specification property.

Corollary

If (X,T) is non-trivial system and T is expansive and has P.O.T.P. and is topologically mixing, then the topological entropy of (X,T) is larger than 0.

Proof: Obvious conclusions from theorem (1) and proposition (3) in [24].

To prove the above theorem we need the following.

Lemma 1.3.

$T:X \rightarrow X$ is topologically mixing if and only if for any $\varepsilon > 0$, there exists an integer $M = M_{\varepsilon} > 0$ such that for all open sets U, V of diameter $\geq \varepsilon$ we have $U \cap T^{-n}V \neq \emptyset$ for all $n \geq M$.

Proof: Given $\varepsilon > 0$, let $\{B_{\alpha}\}_{\alpha \in \Lambda}$ be a finite open cover for X of balls each of which of diameter $\leq \delta$, where δ is taken small enough such that every open set U of diameter $\geq \varepsilon$ should contain at least one of these balls. $\{B_{\alpha}\}_{\alpha \in \Lambda}$ is a finite open cover implies there exists an integer $M > 0$ such that $B_{\alpha} \cap T^{-n}B_{\lambda} \neq \emptyset$ for all $n \geq M$ and for all $\alpha, \lambda \in \Lambda$. The rest follows easily. \square

Definition 1.13. [3].

A homeomorphism $T:X \rightarrow X$ has closed orbit property (C.O.P.) if for any $\varepsilon > 0$ there exist $\delta > 0$, and an integer $N > 0$ such that for any integer $n \geq N$ and every $y \in X$ with $d(T^n y, y) \leq \delta$, there is a point $z \in X$ with $T^n z = z$ (i.e., periodic) such that $d(T^k z, T^k y) \leq \varepsilon$ for all $0 \leq k \leq n$.

Proposition 1.4.

If a homeomorphism $T:X \rightarrow X$ has P.O.T.P. and C.O.P. and topologically mixing, then T has specification property.

Proof: Given $\epsilon > 0$, take $0 < \delta'' < \epsilon/2$ and M_1 from definition of C.O.P., choose δ to satisfy the definition of P.O.T.P. with respect to $0 < \delta' < \delta''/2$ and take M_2 from lemma 1.3 with respect to δ . Let $M = \max(M_1, M_2)$ and let x_0, x_1 be two points in X and $A_0 = [a_0, b_0]$, $A_1 = [a_1, b_1]$ be two strings with $a_1 - b_0 \geq M + 1$. Assume $a_1 - b_0 = n$, so that $n \geq M + 1$. Consider the two open balls $B_\delta(T^{b_0+1}x_0)$, $B_\delta(T^{a_1}x_1)$ ($B_\delta(x)$ is a ball of centre x and radius δ), it is clear $B_\delta(T^{b_0+1}x_0) \cap T^{-(n-1)}B_\delta(T^{a_1}x_1) \neq \emptyset$ for all $n \geq M + 1$.

There exists $x \in X$ such that $d(T^{b_0+1}x_0, x) < \delta$, $d(T^{n-1}x, T^{a_1}x_1) < \delta$.

Also take the two open balls $B_\delta(T^{b_1+1}x_1)$, $B_\delta(T^{a_0}x_0)$, so

$B_\delta(T^{b_1+1}x_1) \cap T^{-(n-1)}B_\delta(T^{a_0}x_0) \neq \emptyset$ for all $n \geq M + 1$.

Also there exists $x' \in X$ such that $d(T^{b_1+1}x_1, x') < \delta$,

$d(T^{n-1}x, T^{a_0}x_0) < \delta$. Now, consider the following sequence of

points $\{T^{a_0}x_0, T^{a_0+1}x_0, \dots, T^{b_0}x_0, x, Tx, \dots, T^{n-2}x, T^{a_1}x_1, \dots, T^{b_1}x_1, x', Tx', \dots, T^{n-2}x', T^{a_0}x_0, T^{a_0+1}x_0, \dots\}$ i.e., define a sequence $\{w_i\}$ as follows.

$$w_i = T^i x_0 \text{ for } a_0 \leq i \leq b_0,$$

$$w_i = T^{i-b_0-1} x \text{ for } b_0 < i < a_1,$$

$$w_i = T^i x_1 \text{ for } a_1 \leq i \leq b_1,$$

$$w_i = T^{i-b_1-1} x' \text{ for } b_1 < i < b_1+n$$

$$w_{b_1+n} = T^{a_0} x_0,$$

and carry on in the same manner. It is obvious that $\{w_i\}$ is a δ -pseudo orbit. There is $z' \in X$ such that

$$d(T^i z', w_i) < \delta' \text{ for } a_0 \leq i \leq b_{1+n}.$$

Hence $d(T^i z', T^i x_0) < \delta' \text{ for } i \in A_0,$

$$d(T^i z', T^i x_1) < \delta' \text{ for } i \in A_1,$$

and

$$d(T^{b_1+n} z', T^{a_0} x_0) < \delta'.$$

Therefore $d(T^{b_1+n} z', T^{a_0} z') < 2\delta' \text{ which implies}$

$$d(T^{b_1-a_0+n} z', T^{a_0} z') < 2\delta' < \delta''.$$

Using C.O.P. there exists $z'' \in X$ such that

$$T^{b_1-a_0+n} z'' = z'' \text{ and } d(T^k z'', T^k T^{a_0} z') < \varepsilon/2 \text{ for } 0 \leq k \leq b_1 - a_0 + n.$$

Let $z = T^{-a_0} z''$, z is periodic of period $b_1 - a_0 + n$ and

$$d(T^{k+a_0} z, T^{k+a_0} z') < \varepsilon/2 \text{ for } 0 \leq i \leq b_1 - a_0 + n \text{ which}$$

implies $d(T^i z, T^i z') < \varepsilon/2 \text{ for } a_0 \leq i \leq b_1 + n.$

The rest follows easily. \square

Using the same idea as proposition (1.4) one can show that if a homeomorphism $T: X \rightarrow X$ satisfies the same hypothesis as the proposition (i.e., T is topologically mixing and has P.O.T.P., C.O.P), then T has the strong specification property (i.e. M is independent of the choice of the number of pieces of orbits. Hence:

Corollary.

If a homeomorphism $T:X \rightarrow X$ is topologically mixing and has P.O.T.P., then T has weak specification property.

The following lemma finishes the proof of theorem 1.

Closed orbit lemma 1.4.

If a homeomorphism $T:X \rightarrow X$ is expansive and has P.O.T.P., then T has closed orbit property. (More over for any $\epsilon > 0$, there exists $\delta > 0$ such that for every positive integer m and for each $x \in X$, if $d(T^m x, x) < \delta$, then there is a periodic point z of period m such that $d(T^i z, T^i x) < \epsilon$ for $0 \leq i \leq m$)

Proof: Take $\epsilon = e(T)$ the expansive constant, take $\delta > 0$ from definition of P.O.T.P. with respect to $\epsilon/2$ and for $m > 0$, assume $d(T^m x, x) < \delta$. Now, consider a sequence $\{y_i\}_{-\infty}^{\infty}$ by taking $y_0 = x$, $y_1 = Tx$, $y_2 = T^2 x, \dots, y_{m-1} = T^{m-1} x$, $y_m = x$, $y_{m+1} = Tx, \dots$, and $y_{-1} = T^{m-1} x$, $y_{-2} = T^{m-2} x, \dots$. Obviously $\{y_i\}_{-\infty}^{\infty}$ is a δ -pseudo orbit, there exists a point $z \in X$ such that $d(T^i z, y_i) < \epsilon/2$ for all $i \in \mathbb{Z}$. But $d(T^i T^m z, y_i) < \epsilon/2$ for all i . Therefore $d(T^i T^m z, T^i z) < \epsilon = e(T)$ for all $i \in \mathbb{Z}$, implies $T^m z = z$. (i.e., z is periodic of period m). \square

CHAPTER II

2.1. PSEUDO ORBIT TRACING PROPERTY.

Let ϕ be a continuous flow on a compact metric space Y the idea is to approximate a chain of orbit segments in Y by one orbit. (i.e., the P.O.T.P. for flows). In [8] Bowen stated that *this idea* is the most important dynamical property, and it was used later in [12], [13], [14], [10]. The idea of defining such property is given also in [12].

Given $\delta, a > 0$ real numbers, a *finite* (δ, a) -chain is a pair of sequences $(\{x_i\}_0^k, \{t_i\}_0^{k-1})$ so that $t_i \geq a$ and $d(\phi_{t_i} x_i, x_{i+1}) < \delta$. An *infinite* (δ, a) -chain is a pair of doubly infinite sequence $(\{x_n\}_{-\infty}^{\infty}, \{t_n\}_{-\infty}^{\infty})$ so that $t_n \geq a$ and $d(\phi_{t_n} x_n, x_{n+1}) < \delta$ for all $n \in \mathbb{Z}$.

The *definition* of a *finite* (*infinite*) (δ, a) -pseudo orbit is the same as a *finite* (*infinite*) (δ, a) -chain, see [12].

Given $\epsilon > 0$, the analogue of ϵ -tracing is given as follows. A finite (*infinite*) (δ, a) -pseudo orbit $(\{x_n\}, \{t_n\})$ is ϵ -traced by the orbit $(\phi_t z)_{t \in \mathbb{R}}$ if there exists an increasing homeomorphism $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ with $\alpha(0) = 0$ such that

$$d[\phi_{\alpha(t)} z, \phi_{t - \sum_{i=0}^{n-1} t_i} x_n] < \epsilon \text{ whenever } t \geq 0, \quad \sum_{i=0}^{n-1} t_i \leq t < \sum_{i=0}^n t_i, \\ n = 0, 1, 2, \dots,$$

$$d[\phi_{\alpha(t)} z, \phi_{t + \sum_{i=-n}^{-1} t_i} x_n] < \epsilon \text{ whenever } t \leq 0, \quad -\sum_{i=-n}^{-1} t_i \leq t < -\sum_{i=-n+1}^{-1} t_i, \\ n = 1, 2, 3, \dots$$

Without loss of generality we are assuming that the subscript (n) of x is positive when $t \geq 0$ and negative when $t < 0$.

A reparameterization $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ of an orbit in an orientation preserving homeomorphism of \mathbb{R} fixing the origin. In the case of finite (δ, a) -pseudo orbit one can restrict α to a closed interval $[a, b]$ containing the origin. As a standard notation when we say (δ, a) -pseudo orbit we mean the infinite (δ, a) -pseudo orbit unless otherwise stated, see [12].

Definition 2.1.

A flow ϕ on Y is said to have the "pseudo orbit tracing property" (P.O.T.P.) if for any $\epsilon > 0$, there is $\delta > 0$ such that each $(\delta, 1)$ -pseudo orbit is ϵ -traced by an orbit of ϕ . (This definition does not depend on the choice of the metric d .)

In this chapter we want to show that the P.O.T.P. for flows is invariant under topological conjugacy with preserved orientation (example, velocity changes), and also the suspension flow of a homeomorphism T under a continuous map f has P.O.T.P. if and only if T has P.O.T.P. In order to be able to prove this we need the following:

Definition 2.2.

A flow ϕ on Y has the pseudo orbit tracing property with respect to time $a > 0$ if for any $\epsilon > 0$, there is $\delta > 0$ such that each (δ, a) -pseudo orbit is ϵ -traced by an orbit of ϕ .

Proposition 2.1.

A flow ϕ on Y has P.O.T.P. with respect to time $a > 0$ if and only if $\forall \epsilon > 0, \exists \delta > 0$ s.t. every (δ, a) -pseudo orbit $(\{x_n\}_{-\infty}^{\infty}, \{\lambda_n\}_{-\infty}^{\infty})$ with $a \leq \lambda_n \leq 2a$ for all $n \in \mathbb{Z}$ is ϵ -traced by an orbit of ϕ .

Proof:

If ϕ has P.O.T.P. with respect to time a , then it is clear that the (δ, a) -pseudo orbit $(\{x_n\}_{-\infty}^{\infty}, \{\lambda_n\}_{-\infty}^{\infty})$ is ϵ -traced by an orbit of ϕ whether $\lambda_n \leq 2a$ or not. Conversely, assume ϕ satisfies the hypothesis of the proposition. Given $\epsilon > 0$, choose $\delta > 0$ using the hypothesis (i.e., every (δ, a) -pseudo orbit $(\{x_n\}, \{\lambda_n\})$ with $a \leq \lambda_n \leq 2a$ is ϵ -traced by an orbit of ϕ .) Let $(\{x_n\}_{-\infty}^{\infty}, \{t_n\}_{-\infty}^{\infty})$ be any (δ, a) -pseudo orbit for ϕ . For each t_n , there is a non-negative integer m_n such that $t_n = a.m_n + r_n$ with $a \leq r_n < 2a$. Construct an infinite sequence of points in Y

$$\{ \dots x_{-3}, \phi_a^{x_{-3}}, \phi_{2a}^{x_{-3}}, \dots, \phi_{a.m_{-3}}^{x_{-3}}, x_{-2}, \phi_a^{x_{-2}}, \phi_{2a}^{x_{-2}}, \dots, \\ \phi_{a.m_{-2}}^{x_{-2}}, x_{-1}, \phi_a^{x_{-1}}, \phi_{2a}^{x_{-1}}, \dots, \phi_{a.m_{-1}}^{x_{-1}}, x_0, \phi_a^{x_0}, \phi_{2a}^{x_0}, \dots, \\ \phi_{a.m_0}^{x_0}, x_1, \phi_a^{x_1}, \phi_{2a}^{x_1}, \dots, \phi_{a.m_1}^{x_1}, x_2, \phi_a^{x_2}, \phi_{2a}^{x_2}, \dots, \phi_{a.m_2}^{x_2}, x_3, \\ \phi_a^{x_3}, \phi_{2a}^{x_3}, \dots, \phi_{a.m_3}^{x_3}, \dots \}$$

(i.e., we can take an infinite sequence $\{y_i\}_{-\infty}^{\infty}$ of points in Y so that

$$y_i = \phi_{ia + \sum_{j=-p}^{-1} a.m_j + p}^{x_{-p}} \text{ for } -\sum_{j=-p}^{-1} m_j - p \leq i \leq -\sum_{j=-p+1}^{-1} m_j - p, p = 1, 2, \dots, \\ y_i = \phi_{ia - \sum_{j=0}^{p-1} a.m_j - p}^{x_p} \text{ for } \sum_{j=0}^{p-1} m_j + p \leq i \leq \sum_{j=0}^p m_j + p, p = 0, 1, 2, \dots)$$

Now also define a sequence $\{\lambda_i\}_{-\infty}^{\infty}$ of real numbers in this way

$$\lambda_i = \begin{cases} a & \text{for } -\sum_{j=-p}^{-1} m_j - p \leq i < -\sum_{j=-p+1}^{-1} m_j - p \\ r_{-p} & \text{for } i = -\sum_{j=-p+1}^{-1} m_j - p, p = 1, 2, 3, \dots, \end{cases} \\ \lambda_i = \begin{cases} a & \text{for } \sum_{j=0}^{p-1} m_j + p \leq i < \sum_{j=0}^p m_j + p \\ r_p & \text{for } i = \sum_{j=0}^p m_j + p, p = 0, 1, 2, \dots \end{cases}$$

which correspond to the sequence $\{y_i\}_{-\infty}^{\infty}$. For $i \geq 0$ assume first

$$\sum_{j=0}^{p-1} m_j + p \leq i < \sum_{j=0}^p m_j + p \text{ and take}$$

$$d(\phi_{\lambda_i} y_i, y_{i+1}) = d(\phi_a \phi_{a.i - \sum_{j=0}^{p-1} a.m_j - p} x_p, \phi_{a.(i+1) - \sum_{j=0}^{p-1} a.m_j - p} x_p) = 0.$$

For $i = \sum_{j=0}^p m_j + p$ we have

$$d(\phi_{\lambda_i} y_i, y_{i+1}) = d(\phi_{r_p} \phi_{a.m_p} x_p, x_{p+1}) = d(\phi_{t_p} x_p, x_{p+1}) < \delta.$$

Also for $i \leq -1$. Assume first $-\sum_{j=-p}^{-1} m_j - p \leq i \leq -\sum_{j=-p+1}^{-1} m_j - p$, then

$$d(\phi_{\lambda_i} y_i, y_{i+1}) = d(\phi_a \phi_{a.i + \sum_{j=-p}^{-1} a.m_j + p} x_p, \phi_{a.(i+1) + \sum_{j=-p}^{-1} a.m_j + p} x_{-p}) = 0$$

and for $i = -\sum_{j=-p+1}^{-1} m_j - p$ we have

$$d(\phi_{\lambda_i} y_i, y_{i+1}) = d(\phi_{r_{-p}} \phi_{a.i + \sum_{j=-p}^{-1} a.m_j + p} x_p, x_{-p+1}) =$$

$$d(\phi_{r_{-p}} \phi_{a.m_{-p}} x_{-p}, x_{-p+1}) = d(\phi_{t_{-p}} x_{-p}, x_{-p+1}) < \delta.$$

So the pair of sequences $(\{y_i\}_{-\infty}^{\infty}, \{\lambda_i\}_{-\infty}^{\infty})$ is (δ, a) -pseudo orbit for ϕ with $a \leq \lambda_i < 2a$. Using the hypothesis of the proposition there is a point $z \in Y$ and a homeomorphism $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ with $\alpha(0) = 0$ such that

$$d(\phi_{\alpha(t)} z, \phi_{t - \sum_{i=0}^{n-1} \lambda_i} y_n) < \varepsilon \text{ for } t \geq 0, \sum_{i=0}^{n-1} \lambda_i \leq t < \sum_{i=0}^n \lambda_i,$$

$$n = 0, 1, 2, \dots,$$

$$d(\phi_{\alpha(t)}^z, \phi_{t+\sum_{i=-n}^{-1} \lambda_i} y_n) < \epsilon \text{ for } t < 0, -\sum_{i=-n}^{-1} \lambda_i \leq t < -\sum_{i=-n+1}^{-1} \lambda_i, n = 1, 2, 3, \dots$$

Now, we can show that the chain $(\{x_n\}_{n=-\infty}^{\infty}, \{t_n\}_{n=-\infty}^{\infty})$ is ϵ -traced

by the orbit $(\phi_t^z)_{t \in \mathbb{R}}$. Assume $i.a \leq t < (i+1).a$, $i \leq m_0$, take

for $0 \leq t < t_0$ we will have

$$\begin{aligned} d(\phi_{\alpha(t)}^z, \phi_{t-t_0} x_0) &= d(\phi_{\alpha(t)}^z, \phi_{t-i.a} \phi_{i.a} x_0) \\ &= d(\phi_{\alpha(t)}^z, \phi_{t-\sum_{j=0}^{i-1} \lambda_j} y_i) < \epsilon, \end{aligned}$$

and for $t_0 \leq t < t_0 + t_1$ we will have

$$d(\phi_{\alpha(t)}^z, \phi_{t-t_0} x_1) = d(\phi_{\alpha(t)}^z, \phi_{t-\sum_{i=0}^{m_0} \lambda_i} x_1).$$

Assume $j.a \leq t - \sum_{i=0}^{m_0} \lambda_i < j.(a+1)$, $j \leq m_1$ implies

$$\sum_{i=0}^{m_0} \lambda_i + \sum_{k=0}^{j-1} \lambda_{m_0+k+1} = \sum_{i=0}^{m_0+j} \lambda_i$$

therefore

$$\begin{aligned} d(\phi_{\alpha(t)}^z, \phi_{t-t_0} x_1) &= d(\phi_{\alpha(t)}^z, \phi_{t-\sum_{i=0}^{m_0+j} \lambda_i} x_1) \\ &= d(\phi_{\alpha(t)}^z, \phi_{t-\sum_{j=0}^{m_0+j} \lambda_j} y_{m_0+j+1}) < \epsilon. \end{aligned}$$

Carry on in the same manner we will have

$$d(\phi_{\alpha(t)}^z, \phi_{k-1}^{x_k}) < \varepsilon \text{ for } t \geq 0, \quad \sum_{i=0}^{k-1} t_i \leq t < \sum_{i=0}^k t_i, \text{ all } k \geq 0.$$

Similarly for $t \leq 0$

$$d(\phi_{\alpha(t)}^z, \phi_{-k}^{x_{-k}}) < \varepsilon \text{ for } -\sum_{i=-k}^{-1} t_i \leq t < -\sum_{i=-k+1}^{-1} t_i, \text{ all } k \geq 1.$$

Hence $(\{x_n\}_{-\infty}^{\infty}, \{\lambda_n\}_{-\infty}^{\infty})$ is ε -traced by the orbit $(\phi_t^z)_{t \in \mathbb{R}}$. So

ϕ has P.O.T.P. with respect to time a . \square

Proposition 2.2.

For every $a > 0$, the flow ϕ on Y has P.O.T.P. with respect to time a if and only if ϕ has P.O.T.P. (i.e., with respect to time 1).

Proof: We have two cases when $a > 1$, $a < 1$. In the case when $a > 1$, assume ϕ has P.O.T.P. with respect to time a . Let m be a natural number $\geq a$. Given $\varepsilon > 0$, choose $\delta > 0$ satisfies the following

- (a) every (δ, a) -pseudo orbit is $\varepsilon/2$ -traced by an orbit of ϕ ,
- (b) $d(x, y) < \delta$ implies $d(\phi_t x, \phi_t y) < \varepsilon/2$ for $0 \leq t \leq 2m$.

Let $\delta' < \delta/m$ and choose $0 < \delta'' < \delta'$ so that $d(x,y) < \delta''$ implies again $d(\phi_t x, \phi_t y) < \delta'$ for $0 \leq t \leq 2m$. Now, let $(\{x_n\}_{n=-\infty}^{\infty}, \{t_n\}_{n=-\infty}^{\infty})$

be a $(\delta'', 1)$ -pseudo orbit for ϕ with $1 \leq t_n \leq 2$ for all $n \in \mathbb{Z}$, take the pair of sequences

$$(\{\dots, x_{-2m}, x_{-m}, x_0, x_m, x_{2m}, \dots\}, \{\dots, \lambda_{-2}, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \dots\})$$

denote $z_i = x_{i.m}$ and $\lambda_i = \sum_{j=0}^{m-1} t_{(j+i.m)}$ for all $i \in \mathbb{Z}$.

Now, take

$$\begin{aligned} d(\phi_{\lambda_i} z_i, z_{i+1}) &= d(\phi_{\sum_{j=0}^{m-1} t_{(j+i.m)}} x_{i.m}, x_{(i+1).m}) \\ &\leq d(\phi_{\sum_{j=1}^{m-1} t_{(j+i.m)}} \phi_{t_{i.m}} x_{i.m}, \phi_{\sum_{j=1}^{m-1} t_{(j+i.m)}} x_{i.m+1}) \\ &\quad + d(\phi_{\sum_{j=2}^{m-1} t_{(j+i.m)}} \phi_{t_{i.m+1}} x_{i.m+1}, \phi_{\sum_{j=2}^{m-1} t_{(j+i.m)}} x_{i.m+2}) + \dots \\ &\quad + d(\phi_{t_{(i+1).m-1}} x_{(i+1).m-1}, x_{(i+1).m}) < \delta' + \delta' + \dots + \delta' \\ &= m\delta' < \delta \text{ because } a \leq \lambda_i \leq 2m. \end{aligned}$$

Therefore $(\{z_i\}_{-\infty}^{\infty}, \{\lambda_i\}_{-\infty}^{\infty})$ is $\alpha(\delta, a)$ -pseudo orbit for ϕ . Hence

there is $z \in Y$ and homeomorphism $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ with $\alpha(0) = 0$

such that

$$d(\phi_{\alpha(t)} z, \phi_{t - \sum_{i=0}^{n-1} \lambda_i} z_n) < \varepsilon/2 \text{ for } t \geq 0, \quad \sum_{i=0}^{n-1} \lambda_i \leq t < \sum_{i=0}^n \lambda_i,$$

$$d(\phi_{\alpha(t)} z, \phi_{t + \sum_{i=-n}^{-1} \lambda_i} z_{-n}) < \varepsilon/2 \text{ for } t \leq 0, \quad -\sum_{i=-n}^{-1} \lambda_i \leq t < -\sum_{i=-n+1}^{-1} \lambda_i.$$

Now, for $0 \leq k \leq m$,

$$d(\phi_{\sum_{i=0}^{k-1} t_i} x_0, x_k) \leq d(\phi_{\sum_{i=1}^{k-1} t_i} \phi_{t_0} x_0, \phi_{\sum_{i=1}^{k-1} t_i} x_1) \\ + d(\phi_{\sum_{i=2}^{k-1} t_i} \phi_{t_1} x_1, \phi_{\sum_{i=2}^{k-1} t_i} x_2) + \dots + d(\phi_{t_{k-1}} x_{k-1}, x_k).$$

Since $\sum_{i=0}^{k-1} t_i \leq 2m$ implies that $d(\phi_{\sum_{i=0}^{k-1} t_i} x_0, x_k) < k\delta' < \delta$.

So

$$d(\phi_{t - \sum_{i=0}^{k-1} t_i + \sum_{i=0}^{k-1} t_i} x_0, \phi_{t - \sum_{i=0}^{k-1} t_i} x_k) < \varepsilon/2 \text{ for } \sum_{i=0}^{k-1} t_i \leq t < \sum_{i=0}^k t_i.$$

Hence

$$d(\phi_{\alpha(t)} z, \phi_{t - \sum_{i=0}^{k-1} t_i} x_k) \leq d(\phi_{\alpha(t)} z, \phi_t z_0) + d(\phi_t z_0, \phi_{t - \sum_{i=0}^{k-1} t_i} x_k) < \varepsilon.$$

Now for $m < k \leq 2m$, also take

$$\begin{aligned}
 d(\phi_{\sum_{i=m}^{k-1} t_i} x_m, x_k) &\leq d(\phi_{\sum_{i=m+1}^{k-1} t_i} \phi_{t_m} x_m, \phi_{\sum_{i=m+1}^{k-1} t_i} x_{m+1}) \\
 &+ d(\phi_{\sum_{i=m}^{k-1} t_i} \phi_{t_{m+1}} x_{m+1}, \phi_{\sum_{i=m+2}^{k-1} t_i} x_{m+2}) + \dots + d(\phi_{\sum_{i=k-1}^{k-1} t_i} x_{k-1}, x_k) \\
 &< (k-m) \delta' < \delta.
 \end{aligned}$$

Now for $\sum_{i=0}^{k-1} t_i \leq t < \sum_{i=0}^k t_i$ implies $t - \sum_{i=0}^{k-1} t_i$ in the closed interval

$[0, 2m]$.

Therefore

$$d(\phi_{t - \sum_{i=0}^{k-1} t_i} x_m, \phi_{t - \sum_{i=0}^{k-1} t_i} x_k) < \varepsilon/2$$

$$\Rightarrow d(\phi_{t - \sum_{i=0}^{k-1} t_i} x_m, \phi_{t - \sum_{i=0}^{k-1} t_i} x_k) < \varepsilon/2$$

$$\Rightarrow d(\phi_{t - \lambda_0} x_m, \phi_{t - \sum_{i=0}^{k-1} t_i} x_k) < \varepsilon/2.$$

Therefore $d(\phi_{\alpha(t)} z, \phi_{t - \sum_{i=0}^{k-1} t_i} x_k) < \varepsilon$ for $\sum_{i=0}^{k-1} t_i \leq t < \sum_{i=0}^k t_i$.

If we carry on in the same manner we will have that the orbit $(\phi_t z)_{t \in \mathbb{R}}$ is ε -traces the (δ, a) -pseudo orbit $(\{x_i\}_{i=-\infty}^{\infty}, \{t_i\}_{i=-\infty}^{\infty})$.

For the other case (when $a < 1$) one can use similar ideas as above. Using proposition 2.1 implies that ϕ has P.O.T.P. \square

2.2. P.O.T.P. AND TOPOLOGICAL CONJUGACY

Using proposition 2.2 we can show

Theorem 2.

If a flow ϕ on X is topologically conjugate to a flow ψ on Y with preserved orientation (see the remark in chapter 1, section 1.3), ~~then either both~~ have P.O.T.P. or ~~neither~~ of them does.

Proof: Assume $\lambda : X \rightarrow Y$ is the conjugating homeomorphism and $\sigma : X \times \mathbb{R} \rightarrow X$ the cocycle of the flow ϕ . Since the map λ preserved orientation, therefore $\sigma_x : \mathbb{R} \rightarrow \mathbb{R}$ is increasing. Let a be the infimum of the set $\{\sigma_x(1); x \in X\}$. By compactness of x and continuity of $\sigma : X \times \mathbb{R} \rightarrow X$ such $a = \inf_{x \in X} \sigma_x(1)$ exists. Now, given $\varepsilon > 0$, choose

$\varepsilon' > 0$ such that $d_Y(y_1, y_2) < \varepsilon'$ implies $d_X(\lambda^{-1}y_1, \lambda^{-1}y_2) < \varepsilon$ for all $y_1, y_2 \in Y$, (where d_X, d_Y are metrics on X, Y respectively.)

Assume ψ has P.O.T.P, using proposition 2.2, there is $\delta' > 0$ such that every (δ', a) -pseudo orbit is ε' -traced by an orbit of ψ .

Also choose $\delta > 0$ such that $d_X(x_1, x_2) < \delta$ implies $d_Y(\lambda x_1, \lambda x_2) < \delta'$ for all $x_1, x_2 \in X$. Now let $(\{x_n\}_{n=-\infty}^{\infty}, \{t_n\}_{n=-\infty}^{\infty})$ be any $(\delta, 1)$ -pseudo orbit for ϕ (i.e., $d_X(\phi_{t_n} x_n, x_{n+1}) < \delta$ for all $n \in \mathbb{Z}$.)

Hence $d_y(\lambda\phi_{t_n} x_n, \lambda x_{n+1}) < \delta'$ for all n . By definition of cocycle

σ we have $\lambda\phi_t x = \psi_{\sigma_x(t)} \lambda x$, therefore

$$d_y(\psi_{\sigma_{x_n}(t_n)} \lambda x_n, \lambda x_{n+1}) < \delta' \text{ for all } n \quad - \quad (a)$$

Consider the pair of sequences $(\{\lambda x_n\}_{n=-\infty}^{\infty}, \{\sigma_{x_n}(t_n)\}_{n=-\infty}^{\infty})$. Since

$t_n \geq 1$ implies $\sigma_{x_n}(t_n) \geq a$ for all $n \in \mathbb{Z}$. Using (a),

$(\{\lambda x_n\}_{n=-\infty}^{\infty}, \{\sigma_{x_n}(t_n)\}_{n=-\infty}^{\infty})$ is a (δ, a) -pseudo orbit for ψ . It follows from

our assumption that ψ has P.O.T.P. that there is $y = \lambda x \in Y$ and

a homeomorphism $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ with $\alpha(0) = 0$ so that

$$d_y(\psi_{\alpha(t)} \lambda x, \psi_{t - \sum_{i=0}^{n-1} \sigma_{x_i}(t_i)} \lambda x_n) < \epsilon' \text{ for } t \geq 0, \quad \sum_{i=0}^{n-1} \sigma_{x_i}(t_i) \leq t < \sum_{i=0}^n \sigma_{x_i}(t_i), \quad (b)$$

$$d_y(\psi_{\alpha(t)} \lambda x, \phi_{t + \sum_{i=-n}^{-1} \sigma_{x_i}(t_i)} \lambda x_{-n}) < \epsilon \text{ for } t \leq 0, \quad -\sum_{i=-n}^{-1} \sigma_{x_i}(t_i) \leq t < -\sum_{i=-n+1}^{-1} \sigma_{x_i}(t_i), \quad (c)$$

(b) implies that

$$d_y(\lambda\phi_{\sigma_x^{-1}\alpha(t)} x, \lambda\phi_{\sigma_{x_n}^{-1}(t - \sum_{i=0}^{n-1} \sigma_{x_i}(t_i))} x_n) < \epsilon' \quad \sum_{i=0}^{n-1} \sigma_{x_i}(t_i) \leq t < \sum_{i=0}^n \sigma_{x_i}(t_i)$$

The way we choose ϵ' we have

$$d_x(\phi_{\sigma_x^{-1}\alpha(t)} x, \phi_{\sigma_{x_n}^{-1}(t - \sum_{i=0}^{n-1} \sigma_{x_i}(t_i))} x_n) < \epsilon \quad \sum_{i=0}^{n-1} \sigma_{x_i}(t_i) \leq t < \sum_{i=0}^n \sigma_{x_i}(t_i).$$

$$\text{For } \sum_{i=0}^{n-1} \sigma_{x_i}(t_i) \leq t < \sum_{i=0}^n \sigma_{x_i}(t_i) \implies$$

$$0 \leq t - \sum_{i=0}^{n-1} \sigma_{x_i}(t_i) < \sigma_{x_n}(t_n) \implies$$

$$0 \leq \sigma_{x_n}^{-1}(t - \sum_{i=0}^{n-1} \sigma_{x_i}(t_i)) < t_n \implies \sum_{i=0}^{n-1} t_i \leq \sigma_{x_n}^{-1}(t - \sum_{i=0}^{n-1} \sigma_{x_i}(t_i)) + \sum_{i=0}^{n-1} t_i < \sum_{i=0}^n t_i$$

Now from (c) we will have also

$$d_x(\phi_{\sigma_x^{-1}\alpha(t)}^x, \phi_{\sigma_{x_n}^{-1}(t + \sum_{i=-n}^{-1} \sigma_{x_i}(t_i))}^{x_{-n}}) < \varepsilon \text{ for } -\sum_{i=-n}^{-1} \sigma_{x_i}(t_i) \leq t < -\sum_{i=-n+1}^{-1} \sigma_{x_i}(t_i)$$

$$\text{Similarly } -\sum_{i=-n}^{-1} \sigma_{x_i}(t_i) \leq t < -\sum_{i=-n+1}^{-1} \sigma_{x_i}(t_i) \implies$$

$$0 \leq t + \sum_{i=-n}^{-1} \sigma_{x_i}(t_i) < \sigma_{x_n}(t_n) \implies$$

$$0 \leq \sigma_{x_n}^{-1}(t + \sum_{i=-n}^{-1} \sigma_{x_i}(t_i)) < t_n \implies$$

$$-\sum_{i=-n}^{-1} t_i \leq \sigma_{x_n}^{-1}(t + \sum_{i=-n}^{-1} \sigma_{x_i}(t_i)) - \sum_{i=-n}^{-1} t_i < -\sum_{i=-n+1}^{-1} t_i.$$

Now define $h: \mathbb{R} \rightarrow \mathbb{R}$ in this way, for $t \geq 0$ let

$$u = \sigma_{x_n}^{-1}(t - \sum_{i=0}^{n-1} \sigma_{x_i}(t_i)) + \sum_{i=0}^{n-1} t_i, \text{ and take } h(u) = \sigma_x^{-1}\alpha(t),$$

$$\text{and, for } t \leq 0, \text{ let } v = \sigma_{x_n}^{-1}(t + \sum_{i=-n}^{-1} \sigma_{x_i}(t_i)) - \sum_{i=-n}^{-1} t_i \text{ and take}$$

$h(v) = \sigma_x^{-1}\alpha(t)$. It is obvious that h is continuous and increasing

with $h(0) = 0$. Finally for $\sum_{i=0}^{n-1} t_i \leq u < \sum_{i=0}^n t_i$. We have

$$d_x(\phi_{h(u)}x, \phi_{u - \sum_{i=0}^{n-1} t_i} x_n) < \epsilon,$$

for $-\sum_{i=-n}^{-1} t_i \leq u < -\sum_{i=-n+1}^{-1} t_i$ we have

$$d_x(\phi_{h(u)}x, \phi_{u + \sum_{i=-n}^{-1} t_i} x_n) < \epsilon.$$

Therefore $(\{x_n\}_{-\infty}^{\infty}, \{t_n\}_{-\infty}^{\infty})$ is ϵ -traced by an orbit $(\phi_t x)_{t \in \mathbb{R}}$. \square

Corollary

If a flow ϕ on X is obtained from a flow ψ on X by changing velocity and ψ has P.O.T.P, then so does ϕ .

Remark.

Since any two suspension flows of a given homeomorphism $T:X \rightarrow X$ are topologically conjugate with preserved orientation, so for this reason we shall concentrate on suspensions under the constant function 1, (see section 1.3 in chapter one.)

Lemma 2.1.

Let ϕ on Y be the suspension flow of a homeomorphism $T:X \rightarrow X$ under a constant map 1. (i.e., $X_1 = Y$, $\phi = T_1$.) Then for every

$\lambda > 0$, there is $\lambda' > 0$ such that for every two points $(x,s), (y,t) \in Y$ with $\rho((x,s), (y,t)) < \lambda'$, then either $d(x,y) < \lambda$ or $d(Tx,y) < \lambda$ or $d(T^{-1}x,y) < \lambda$. (where d is a metric on X and ρ is the metric we define in section (1.3) Chapter 1 on the suspension space $Y = X_1$).

Proof: Take $0 < \lambda'' < \lambda$, $\lambda'' < \frac{1}{4}$ and so that $d(x,y) < \lambda''$ implies $d(T^i x, T^i y) < \lambda$ for $i = -1, 0, 1$. Now let $\lambda' = \frac{\lambda''}{2}$ and assume $\rho((x,s), (y,t)) < \lambda'$. Using Lemma (1.2) we will have either $|s-t| < \lambda'$ or $|1+s-t| < \lambda'$ or $|1+t-s| < \lambda'$. Now, for the first case when $|s-t| < \lambda'$ we will have $\rho((x,s), (y,s)) \leq \rho((x,s), (y,t)) + \rho((y,t), (y,s)) < \lambda' + \lambda' = \lambda''$. Hence $(1-s)d(x,y) + sd(Tx,Ty) < \lambda''$. So $d(x,y) < \lambda''$ or $d(Tx,Ty) < \lambda''$ implies that $d(x,y) < \lambda$. For the case when $|1+t-s| < \lambda''$ we will have $\rho((Tx,t), (y,t)) \leq \rho((Tx,t), (x,s)) + \rho((x,s), (y,t)) < \lambda' + \lambda' = \lambda''$. Hence $(1-t)d(Tx,y) + td(T^2x,Ty) < \lambda''$ implies that $d(Tx,y) < \lambda$. For the case when $|1+s-t| < \lambda''$ we will have $\rho((T^{-1}x,t), (y,t)) \leq \rho((T^{-1}x,t), (x,s)) + \rho((x,s), (y,t)) < \lambda' + \lambda' = \lambda''$. Hence $(1-t)d(T^{-1}x,y) + td(x,Ty)$ implies that $d(T^{-1}x,y) < \lambda$. \square

2.3. THE SUSPENSION FLOW

To complete the *study* of the pseudo orbit tracing property for flows, we are going to close this chapter with the following:

Theorem 3.

Let T be a homeomorphism of (X, d) , and $f: Y \rightarrow \mathbb{R} > 0$ be continuous. The suspension flow of T has pseudo orbit tracing property if and only if T has pseudo orbit tracing property.

Proof: By our earlier remark we need only to show the results when f is constant map 1. Let $\phi: Y \times \mathbb{R} \rightarrow Y$ denote the suspension of T under 1. Suppose ϕ has P.O.T.P. Given $\epsilon > 0$, take $0 < \epsilon' < \epsilon$, $\epsilon' < \frac{1}{4}$ and $d(x, y) < \epsilon'$ implies $d(T^i x, T^i y) < \epsilon$ for $i = -1, 0, 1$. Choose $\delta > 0$ from definition of P.O.T.P. for ϕ with respect to ϵ' . Also take $0 < \delta' < \delta$ so that $d(x, y) < \delta'$ implies $d(Tx, Ty) < \delta$. Now let $\{x_n\}_{-\infty}^{\infty}$ be any δ' -pseudo orbit for T . (i.e., $d(Tx_n, x_{n+1}) < \delta'$ for all $n \in \mathbb{Z}$). Consider the pair of sequences $(\{(x_n, \frac{1}{2})\}_{-\infty}^{\infty}, \{t_n\}_{-\infty}^{\infty})$ so that $t_n = 1$ for all $n \in \mathbb{Z}$.

$$\rho(\phi_1(x_n, \frac{1}{2}), (x_{n+1}, \frac{1}{2})) = \rho((Tx_n, \frac{1}{2}), (x_{n+1}, \frac{1}{2})) = \frac{1}{2} d(Tx_n, x_{n+1}) + \frac{1}{2} d(T^2 x_n, Tx_{n+1}) < \delta.$$

So $(\{(x_n, \frac{1}{2})\}_{-\infty}^{\infty}, \{t_n = 1\}_{-\infty}^{\infty})$ is $(\delta, 1)$ -pseudo orbit for ϕ . Using the definition of P.O.T.P. implies that there exists $(x, s) \in Y$ and a homeomorphism $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ with $\alpha(0) = 0$ such that

$$\rho(\phi_{\alpha(t)}(x, s), \phi_{t-n}(x_n, \frac{1}{2})) < \epsilon' \text{ for } t \geq 0, n \leq t < n+1,$$

$$\rho(\phi_{\alpha(t)}(x, s), \phi_{t+n}(x_{-n}, \frac{1}{2})) < \epsilon' \text{ for } t < 0, -n \leq t < -n+1$$

Now as $t = 0$ we have $\rho((x, s), (x_0, \frac{1}{2})) < \epsilon' < \frac{1}{4}$, so $|s - \frac{1}{2}| < \epsilon'$.

Also $\rho(\phi_{\alpha(1)}(x,s), (x_1, \frac{1}{2})) < \varepsilon' < \frac{1}{4}$ and since

$$\rho(\phi_{\alpha(t)}(x,s), \phi_t(x_0, \frac{1}{2})) < \varepsilon' < \frac{1}{4} \text{ for } 0 \leq t < 1,$$

so $\phi_{\alpha(1)}(x,s)$ should be represented as $(Tx, s') \in Y$, where $0 \leq s' < 1$,

$$|s' - \frac{1}{2}| < \varepsilon'. \text{ Also we have } \rho(\phi_{\alpha(t)}(x,s), \phi_{t-1}(x_1, \frac{1}{2})) < \varepsilon' < \frac{1}{4}$$

for all $1 \leq t < 2$ and $\rho(\phi_{\alpha(2)}(x,s), (x_2, \frac{1}{2})) < \varepsilon' < \frac{1}{4}$, so from

continuity of α also $\phi_{\alpha(2)}(x,s)$ should be represented as (T^2x, S'') ,

where $0 \leq S'' < 1, |S'' - \frac{1}{2}| < \varepsilon'$. If we carry on in the same manner

we will have $\phi_{\alpha(n)}(x,s)$ should be as $(T^n x, S^{(n)})$, where

$0 \leq S^{(n)} < 1, |S^{(n)} - \frac{1}{2}| < \varepsilon'$ for all $n = 0, 1, 2, \dots$, and $\phi_{\alpha(-n)}(x,s)$

should be represented as $(T^{-n}x, S^{(-n)})$, where $0 \leq S^{(-n)} < 1$,

$$|S^{(-n)} - \frac{1}{2}| < \varepsilon' \text{ for } n = 1, 2, \dots. \text{ So}$$

$$\rho((T^n x, S^{(n)}), \phi_{t-n}(x_n, \frac{1}{2})) < \varepsilon' \text{ for } n \leq t < n+1, \text{ all } n \in \mathbb{Z}.$$

For $t = n$ implies $\rho((T^n x, S^{(n)}), (x_n, \frac{1}{2})) < \varepsilon' < \frac{1}{4}$. But

$$\rho((T^n x, \frac{1}{2}), (x_n, \frac{1}{2})) \leq \rho((T^n x, S^{(n)}), (x_n, \frac{1}{2})) < \varepsilon' < \frac{1}{4}, \text{ therefore}$$

$$\frac{1}{2}d(T^n x, x_n) + \frac{1}{2}d(T^{n+1}x, Tx_n) < \varepsilon'. \text{ Hence } d(T^n x, x_{n+1}) < \varepsilon' \text{ or}$$

$$d(T^{n+1}x, Tx_n) < \varepsilon'. \text{ From the way we choose } \varepsilon' \text{ implies that}$$

$$d(T^n x, x_n) < \varepsilon \text{ for all } n \in \mathbb{Z}. \text{ Therefore } T \text{ has P.O.T.P.}$$

Conversely, assume T has P.O.T.P. Given $\varepsilon > 0$, take $0 < \varepsilon' < \varepsilon$

so that $d(x, y) < \varepsilon'$ implies $d(T^i x, T^i y) < \varepsilon/2$ for $i = 0, 1, 2$. Let

$0 < \delta < \varepsilon'/2$ and δ satisfies the definition of P.O.T.P. for T with

respect to ε' . Take $0 < \delta' < \delta$ as in lemma 2.1. Assume

$(\{(x_k, s_k)_{-\infty}^{\infty}, \{t_k\}_{-\infty}^{\infty})$ is a $(\delta', 2)$ -pseudo orbit for the suspension flow ϕ , (i.e. $\rho(\phi_{t_k}(x_k, s_k), (x_{k+1}, s_{k+1})) < \delta'$ for all $k \in \mathbb{Z}$.) where $0 \leq s_k < 1$. (i.e., $(x_k, s_k) \in Y$ for all $k \in \mathbb{Z}$.) Let $w_k = [s_k + t_k]$ the integral part of $s_k + t_k$. (i.e., w_k is a largest integer $\leq s_k + t_k$.) Hence

$$\rho((T^{w_k} x_k, s_k + t_k - w_k), (x_{k+1}, s_{k+1})) < \delta' \text{ for all } k \in \mathbb{Z}.$$

Using lemma 1.2, 2.1 implies that either $|s_k + t_k - w_k - s_{k+1}| < \delta'$ or $|1 + s_k + t_k - w_k - s_{k+1}| < \delta'$ or $|1 + s_{k+1} - s_k - t_k + w_k| < \delta'$. Now, let n_k be a positive integer defined as

$$n_k = \begin{cases} w_k & \text{if } |s_k + t_k - w_k - s_{k+1}| < \delta' \\ w_k - 1 & \text{if } |1 + s_k + t_k - w_k - s_{k+1}| < \delta' \\ w_k + 1 & \text{if } |1 + s_{k+1} - s_k - t_k + w_k| < \delta', \end{cases}$$

so it is obvious that in each of the above cases we will have

$|s_k + t_k - n_k - s_{k+1}| < \delta'$ for all $k \in \mathbb{Z}$. Since $t_k \geq 2$ for all $k \in \mathbb{Z}$ implies $n_k \geq 1$ for all $k \in \mathbb{Z}$. Using lemma 2.1 we will have for every above case $d(T_{x_k}^{n_k}, x_{k+1}) < \delta$ for all $k \in \mathbb{Z}$.

Define a sequence $\{y_i\}_{-\infty}^{\infty}$ in X so that

$$y_i = T^i x_0 \quad \text{for } 0 \leq i < n_0$$

$$y_i = T^{i-n_0} x_1 \quad \text{for } n_0 \leq i < n_0 + n_1$$

$$\begin{aligned} & \vdots \\ & \vdots \quad k-1 \\ & y_i = T^{i - \sum_{j=0}^{k-1} n_j} x_k \quad \text{for } \sum_{j=0}^{k-1} n_j \leq i < \sum_{j=0}^k n_j \\ & \vdots \end{aligned}$$

$$y_i = T^{n_{-1}^{-i}} x_{-1} \quad \text{for } -n \leq i < -1$$

$$y_i = T^{n_{-1} + n_{-2}^{-i}} x_{-2} \quad \text{for } -n_{-1} - n_{-2} \leq i < -n_{-1},$$

and carry on, then obviously that this sequence is δ -pseudo orbit for ϕ . (i.e., $d(Ty_i, y_{i+1}) < \delta$ for all $i \in \mathbb{Z}$.) T has P.O.T.P.

implies there exists $x \in X$ such that $d(T^k x, x_k) < \epsilon'$ for all $k \in \mathbb{Z}$.

Hence $d(T^{j + \sum_{i=0}^{k-1} n_i} x, T^j x_k) < \epsilon'$ for $0 \leq j \leq n_k$, $k = 0, 1, 2, \dots$ and for negative indices

$$d(T^{j - \sum_{i=-k}^{-1} n_i} x, T^j x_{-k}) < \epsilon' \quad \text{for } 0 \leq j < n_{-1} - 1, k = 1, 2, \dots$$

Now, take the point $(x, s_0) \in Y$ and define $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ in the following way.

For $t \geq 0$, let

$$\alpha(t) = \frac{s_1 + n_0 - s_0}{t_0} t \quad \text{for } 0 \leq t < t_0$$

$$\alpha(t) = \frac{s_2 + n_1 - s_1}{t_1} (t - t_0) + s_1 + n_0 - s_0 \quad \text{for } t_0 \leq t < t_0 + t_1$$

$$\alpha(t) = \frac{s_{k+1} + n_k - s_k}{t_k} \left(t - \sum_{i=0}^{k-1} t_i \right) + s_k + \sum_{i=0}^{k-1} n_i - s_0 \text{ for } \sum_{i=0}^{k-1} t_i \leq t < \sum_{i=0}^k t_i$$

and for $t \leq 0$, let

$$\alpha(t) = \frac{s_{-1} - s_0 - n_{-1}}{-t_{-1}} t \text{ for } -t_{-1} \leq t < 0$$

$$\alpha(t) = \frac{s_{-2} - s_{-1} - n_{-2}}{-t_{-2}} (t + t_{-1}) + s_{-1} - s_0 - n_{-1} \text{ for } -t_{-1} - t_{-2} \leq t < -t_{-1}$$

$$\alpha(t) = \frac{s_{-k} - s_{-k+1} - n_{-k}}{-t_{-k}} \left(t + \sum_{i=-k+1}^{-1} t_i \right) + s_{-k+1} - s_0 - \sum_{i=-k+1}^{-1} n_i \text{ for } -\sum_{i=-k}^{-1} t_i \leq t < -\sum_{i=-k+1}^{-1} t_i$$

It is clear that α is continuous with $\alpha(0) = 0$ and because $n_k \geq 1$ α becomes increasing homeomorphism. We claim that $(\phi_t(x, s_0))_{t \in \mathbb{R}}$ is an orbit which ε -traces $(\{(x_k, s_k)\}_{-\infty}^{\infty}, \{t_k\}_{-\infty}^{\infty})$. Pick $t \geq 0$ and

without loss of generality assume

$$\sum_{i=0}^{k-1} t_i \leq t < \sum_{i=0}^k t_i \text{ and take}$$

$$\begin{aligned} |s_0 + \alpha(t) - \sum_{i=0}^{k-1} n_i - s_k - t + \sum_{i=0}^{k-1} t_i| &= \left| \frac{s_{k+1} + n_k - s_k}{t_k} \left(t - \sum_{i=0}^{k-1} t_i \right) + s_k - s_k - \left(t - \sum_{i=0}^{k-1} t_i \right) \right| \\ &= \left| \frac{s_{k+1} + n_k - s_k - t_k}{t_k} \left(t - \sum_{i=0}^{k-1} t_i \right) \right| = |s_{k+1} + n_k - s_k - t_k| \cdot \left| \frac{t - \sum_{i=0}^{k-1} t_i}{t_k} \right|. \end{aligned}$$

$$\text{Since } |s_{k+1} + n_k - s_k - t_k| < \delta', \quad \left| \frac{t - \sum_{i=0}^{k-1} t_i}{t_k} \right| < 1,$$

therefore $|s_0 + \alpha(t) - \sum_{i=0}^{k-1} n_i - s_k - t + \sum_{i=0}^{k-1} t_i| < \delta'.$

Now, since $w_k = [s_k + t_k]$, $t - \sum_{i=0}^{k-1} t_i < t_k$ and if j an integer

which makes $0 \leq s_k + t - \sum_{i=0}^{k-1} t_i - j < 1$, then $0 \leq j \leq n_k + 1$.

so $d(T^{\sum_{i=0}^{k-1} n_i} x, T^j x_k) < \epsilon/2$ for $0 \leq j \leq n_k + 2$. Finally, take

$$\begin{aligned} & \rho((T^{\sum_{i=0}^{k-1} n_i} x, s_0 + \alpha(t) - \sum_{i=0}^{k-1} n_i), (x_k, s_k + t - \sum_{i=0}^{k-1} t_i)) \\ &= \rho((T^{\sum_{i=0}^{k-1} n_i} x, s_0 + \alpha(t) - \sum_{i=0}^{k-1} n_i - j), (x_k, s_k + t - \sum_{i=0}^{k-1} t_i - j)) \\ &\leq \rho((T^{\sum_{i=0}^{k-1} n_i} x, s_0 + \alpha(t) - \sum_{i=0}^{k-1} n_i - j), (T^{\sum_{i=0}^{k-1} n_i} x, s_k + t - \sum_{i=0}^{k-1} t_i - j)) \\ &+ \rho((T^{\sum_{i=0}^{k-1} n_i} x, s_k + t - \sum_{i=0}^{k-1} t_i - j), (T^j x_k, s_k + t - \sum_{i=0}^{k-1} t_i - j)) \\ &\leq |s_0 + \alpha(t) - \sum_{i=0}^{k-1} n_i - j - s_k - t + \sum_{i=0}^{k-1} t_i + j| + (1 - s_k - t + \sum_{i=0}^{k-1} t_i + j) \\ &\quad d(T^{\sum_{i=0}^{k-1} n_i} x, T^j x_k) + (s_k + t - \sum_{i=0}^{k-1} t_i - j) d(T^{\sum_{i=0}^{k-1} n_i} x, T^{j+1} x_k) \\ &< \delta' + (1 - s_k - t + \sum_{i=0}^{k-1} t_i + j)\epsilon/2 + (s_k + t - \sum_{i=0}^{k-1} t_i - j)\epsilon/2 \leq \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Therefore, for $t \geq 0$, $\sum_{i=0}^{k-1} t_i \leq t < \sum_{i=0}^k t_i$ we proved that

$$\rho(\phi_{\alpha(t)}(x, s_0), \phi_{t - \sum_{i=0}^{k-1} t_i}(x_k, s_k)) < \varepsilon,$$

and for $t \leq 0$, similar as above we can show

$$\rho(\phi_{\alpha(t)}(x, s_0), \phi_{t + \sum_{i=-k}^{-1} t_i}(x_{-k}, s_{-k})) < \varepsilon \text{ for } -\sum_{i=-k}^{-1} t_i \leq t < -\sum_{i=-k+1}^{-1} t_i.$$

Therefore ϕ has P.O.T.P. \square

CHAPTER III

In this chapter we are going to show that a flow ϕ which has P.O.T.P. is topologically stable under the extra property of "expansiveness"

Let ϕ be a continuous flow on a compact metric space (Y, d) .

Definition 3.1. [2]

ϕ is expansive if for any $\epsilon > 0$ there is $\delta > 0$ with the property that if $x, y \in Y$ and $s: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous map with $s(0) = 0$ so that $d(\phi_t x, \phi_{s(t)} y) < \delta$ for all $t \in \mathbb{R}$, then $y = \phi_t x$, with $|t| < \epsilon$.

(This definition does not depend on the choice of metric on Y .)

Lemma 3.1. (Cf [2] lemma 1)

If ϕ is an expansive flow, then each fixed point of ϕ is an isolated point.

This lemma shows the study of expansive flows can be reduced to those without fixed points.

Theorem 3.2. (Cf. [2] theorem 3(iv))

Let ϕ be a continuous flow on Y without fixed points. Then ϕ is expansive if and only if for any $\epsilon > 0$, there is $r > 0$ such that if $t = (t_i)_{i=-\infty}^{\infty}$ and $u = (u_i)_{i=-\infty}^{\infty}$ are doubly infinite sequences

of real numbers with $u_0 = t_0 = 0$, $0 < t_{i+1} - t_i \leq r$, $|u_{i+1} - u_i| \leq r$, $t_i \rightarrow \infty$, $t_i \rightarrow -\infty$ as $i \rightarrow \infty$, and if $x, y \in Y$ satisfies $d(\phi_{t_i} x, \phi_{u_i} y) \leq r$ for all $i \in \mathbb{Z}$, then $y = \phi_t x$ with $|t| < \varepsilon$.

Lemma 3.3. (Cf.[2], lemma 2)

If a flow ϕ has no fixed points, there is $T_0 > 0$ such that for every T satisfying $0 < T < T_0$ there exists $\gamma > 0$ with $d(\phi_T x, x) \geq \gamma$ for all $x \in Y$. (In fact T_0 is the smallest positive number with $\phi_{T_0} x = x$, and if ϕ has no periodic points, $T_0 = 1$, see [2].)

Also we need the following.

Lemma 3.4.

Let ϕ be a continuous flow on Y . Then for every $\eta, T > 0$ there is $\lambda > 0$ such that if $d(y_1, y_2) < \lambda$, then $d(\phi_t y_1, \phi_t y_2) < \eta$ for all $t \in [-T, T]$, (where $[-T, T]$ is a closed interval in \mathbb{R} .)

Proof: Assume the above is not true, there exist $\eta, T > 0$ such that if $\{\lambda_i\}_{i=0}^{\infty}$ is a sequence of positive real number with $\lambda_i \rightarrow 0$, then for every i , there exists $y_i, z_i \in Y$, $t_i \in [-T, T]$ so that $d(y_i, z_i) < \lambda_i$ but $d(\phi_{t_i} y_i, \phi_{t_i} z_i) \geq \eta$. By compactness of Y , $[-T, T]$, without loss of generality one can assume $y_i \rightarrow y$, $z_i \rightarrow z$ and $t_i \rightarrow t$ where $y, z \in Y$, $t \in [-T, T]$. $d(y_i, z_i) < \lambda_i$ for all i implies that $d(y, z) = 0$ which implies $y = z$. And $d(\phi_{t_i} z_i, \phi_{t_i} y_i) \geq \eta$ for all i implies $d(\phi_t z, \phi_t y) \geq \eta > 0$ and this is a contradiction. \square

Lemma 3.5.

Let $\{\alpha_j\}_{j=1}^{\infty}$ be a family of continuous increasing functions from $[0,a]$ into R with $\alpha_j(0) = 0$ for all j and assume $\alpha_j(a) \rightarrow \infty$ as $j \rightarrow \infty$. Then for every $\lambda, \beta > 0$, there are j and $t_1, t_2 \in [0,a]$ with $t_1 < t_2$ such that $t_2 - t_1 < \lambda$ and $\alpha_j(t_2) - \alpha_j(t_1) = \beta$ (where $[0,a] \subseteq R$ is a closed interval.)

Proof: Let $\{s_k\}_{k=0}^n$ be a partition of $[0,a]$ with mesh $< \lambda$ (i.e. s_k are elements in $[0,a]$ such that $s_0 = 0 < s_1 < s_2 < \dots < s_n = a$ with $s_{k+1} - s_k < \lambda$ for $0 \leq k \leq n-1$.) and assume for every $j, \alpha_j(s_{k+1}) - \alpha_j(s_k) < \beta$. Then $\alpha_j(s_n) - \alpha_j(s_0) < n\beta$ for all j . But $\alpha_j(s_n) - \alpha_j(s_0) = \alpha_j(a) \rightarrow \infty$ as $j \rightarrow \infty$, this is a contradiction. Therefore there exists a positive integer N such that for every $j \geq N$, there are $s_k, s_{k+1} \in [0,a]$ so that $\alpha_j(s_{k+1}) - \alpha_j(s_k) \geq \beta$. By continuity of α_j if we take $t_2 = s_{k+1}$ we can choose $t_1 \in [s_k, s_{k+1}]$ such that $\alpha_j(t_2) - \alpha_j(t_1) = \beta$. \square

Corollary.

Let $\{\alpha_j\}_{j=1}^{\infty}$ be a family of continuous increasing functions from $[a,0]$ into R (i.e., here $a < 0$) with $\alpha_j(0) = 0$ for all j and assume $\alpha_j(a) \rightarrow -\infty$ as $j \rightarrow \infty$. Then for every $\lambda, \beta > 0$, there are j and $t_1, t_2 \in [a,0]$ with $t_1 < t_2$ such that $t_2 - t_1 < \lambda$ and $\alpha_j(t_2) - \alpha_j(t_1) = \beta$.

Proof: Exactly similar to the above.

Let ϕ, ψ be two continuous flows on Y , define

$$d(\phi_t, \psi_t) = \sup_{x \in X} d(\phi_t x, \psi_t x)$$

Definition 3.2.

A flow ϕ on Y is said to be topologically stable if for any $\epsilon > 0$ there is $\delta > 0$ such that for every other flow ψ on Y with $d(\phi_t, \psi_t) < \delta$ for all $t \in [0,1]$, there exists $h: Y \rightarrow Y$ continuous such that $d(h, id) < \epsilon$ and $h(\text{orbit of } \psi) \subseteq \text{orbit of } \phi$. (This definition does not depend on the choice of metric)

By using the above lemmas we are going to prove the following main theorem.

Theorem 4.

Every continuous expansive flow on Y without fixed points which has pseudo orbit tracing property is topologically stable.

Proof: Given $\epsilon > 0$ and without loss of generality assume

$0 < \epsilon < T_0/2$ (where T_0 as in lemma 3.3). Using Theorem 3.2.

take $0 < r < \epsilon$. Now since $0 < r < T_0/2$, using lemma 3.3, there is $\gamma > 0$ so that $d(\phi_r y, y) \geq \gamma$ for all $y \in Y$. From definition

of expansiveness take $\epsilon' < \gamma$, $\epsilon' < r$ and if $d(\phi_{s(t)} x, \phi_t y) \leq \epsilon'$ for all $x, y \in Y$ and a continuous map $s: \mathbb{R} \rightarrow \mathbb{R}$ with $s(0) = 0$, then $y = \phi_t x$ with $|t| \leq r$. Also choose $0 < \delta < \epsilon'/12$ so that

- (a) every $(\delta, 1)$ -pseudo orbit is $\varepsilon'/12$ -traced by an orbit of ϕ ,
 (b) for every $x, y \in Y$ $d(x, y) < \delta$ implies $d(\phi_t x, \phi_t y) < \varepsilon'/12$ for all t in the closed interval $[0, 1]$.

Assume ψ be another continuous flow on Y with $d(\phi_t, \psi_t) < \delta$ for all $t \in [0, 1]$ ($d(\phi_t, \psi_t) = \sup_{x \in Y} d(\phi_t x, \psi_t x)$) and fix a point $y \in Y$.

Since $d(\phi_1 \psi_n(y), \psi_{n+1}(y)) < \delta$ for all $n \in \mathbb{Z}$, therefore

$(\{\psi_n(y)\}_{n=-\infty}^{\infty}, \{t_n = 1\}_{n=-\infty}^{\infty})$ is a $(\delta, 1)$ -pseudo orbit for ϕ . Using

definition of P.O.T.P. there exists $z \in Y$ and $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ increasing homeomorphism with $\alpha(0) = 0$ such that for $t \geq 0$

$d(\phi_{\alpha(t)} z, \psi_{t-n}(y)) < \varepsilon'/12$ whenever $n \leq t < n+1$, $n = 0, 1, 2, \dots$,
 for $t \leq 0$

$d(\phi_{\alpha(t)} z, \psi_{t+n}(y)) < \varepsilon'/2$ whenever $-n \leq t < -n+1$ $n = 1, 2, 3, \dots$.

For the case when $t \geq 0$ assume m, k are positive integers such that $n \leq \frac{m}{k} \leq t < \frac{m+1}{k} \leq n+1$, so

$$d(\psi_{\frac{m}{k}-n}(y), \phi_{\frac{m}{k}-n}(y)) < \delta$$

and from (b) above we will have

$$\begin{aligned} d(\phi_{t-\frac{m}{k}} \psi_{\frac{m}{k}}(y), \phi_{t-n} \psi_n(y)) &= d(\phi_{t-\frac{m}{k}} \psi_{\frac{m}{k}}(y), \phi_{t-\frac{m}{k}} \phi_{\frac{m}{k}-n} \psi_n(y)) \\ &= d(\phi_{t-\frac{m}{k}} \psi_{\frac{m}{k}-n}(y), \phi_{t-\frac{m}{k}} \phi_{\frac{m}{k}-n} \psi_n(y)) < \varepsilon'/12. \end{aligned}$$

Therefore $d(\phi_{\alpha(t)}^z, \phi_{t-\frac{m}{k}}^{\psi_{\frac{m}{k}}(y)}) < \varepsilon'/6$ whenever $\frac{m}{k} \leq t < \frac{m+1}{k}$.

As we know for every $t \in \mathbb{R}$ there is a sequence of rational numbers $\{\frac{m_i}{t_i}\}$ such that $\frac{m_i}{t_i} \rightarrow t$ so it follows from above that

$$d(\phi_{\alpha(t)}^z, \phi_{t-\frac{m_i}{t_i}}^{\psi_{\frac{m_i}{t_i}}(y)}) < \varepsilon'/6 \text{ for all } i.$$

Similar ideas can be carried on when $t \leq 0$, so this means that we proved here for an orbit $(\psi_t y)_{t \in \mathbb{R}}$ there exists a point $z \in Y$ and an increasing homeomorphism $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ with $\alpha(0) = 0$ such that

$$(i) \quad d(\phi_{\alpha(t)}^z, \psi_t y) \leq \varepsilon'/6 \text{ for all } t \in \mathbb{R}.$$

Assume $z' \in Y$ and $\alpha': \mathbb{R} \rightarrow \mathbb{R}$ increasing homeomorphism with $\alpha'(0) = 0$ and such that we have also

$$d(\phi_{\alpha'(t)}^{z'}, \psi_t(y)) \leq \varepsilon'/6 \text{ for all } t \in \mathbb{R}.$$

Then $d(\phi_{\alpha(t)}^z, \phi_{\alpha'(t)}^{z'}) \leq \varepsilon'/3$ for all $t \in \mathbb{R}$. Using expansiveness and the way we choose ε' implies $z' = \phi_t z$ with $|t| \leq r < \varepsilon$. It means that we proved every orbit of ψ is $\varepsilon'/6$ -traced by a unique orbit of ϕ .

Now for $y \in Y$ define the set A_y as follows

$A_y = \{x \in Y: \text{for every } \eta, T > 0, \text{ there exists a homeomorphism } \alpha: \mathbb{R} \rightarrow \mathbb{R} \text{ with } \alpha(0) = 0 \text{ such that}$

$$d(\phi_{\alpha(t)}^x, \psi_t y) < \varepsilon'/6 + \eta \text{ for all } t \in [-T, T]\}.$$

Now from (i) above we know that $(\psi_t y)_{t \in \mathbb{R}}$ is uniquely traced by an orbit of ϕ say $(\phi_t z)_{t \in \mathbb{R}}$, we want to show

$$(1) \quad A_y \subseteq (\phi_t z)_{t \in \mathbb{R}} \text{ and time diameter of } A_y < \varepsilon,$$

$$(2) \quad A_y \text{ is a closed set in } Y.$$

To prove (1) as we know $z \in A_y$, (i.e., $d(\phi_{\alpha(t)} z, \psi_t y) \leq \varepsilon'/6$ for all $t \in \mathbb{R}$) assume $\{\eta_i\}_0^\infty, \{T_i\}_0^\infty$ are sequences of positive real numbers such that $\eta_i \rightarrow 0, T_i \rightarrow \infty$ as $i \rightarrow \infty$. If $x \in A_y$, there are $\alpha_i: \mathbb{R} \rightarrow \mathbb{R}$ homeomorphisms with $\alpha_i(0) = 0$ such that

$$d(\phi_{\alpha_i(t)} x, \psi_t y) < \varepsilon'/6 + \eta_i \text{ for all } t \in [-T_i, T_i].$$

Using (i) we have

$$d(\phi_{\alpha_i(t)} x, \phi_{\alpha(t)} z) < \varepsilon'/3 + \eta_i \text{ for } t \in [-T_i, T_i].$$

Since $\eta_i \rightarrow 0$, without loss of generality assume $\eta_i < \varepsilon'/6$

for all i . Take $T'_i = \min \{(|\alpha(T_i)|, |\alpha(-T_i)|)\}; (\min\{a, b\} = \text{minimum of } a, b)$. It is clear that $T'_i \rightarrow \infty$ as $i \rightarrow \infty$, so

$$d(\phi_{\alpha_i \alpha^{-1}(u)} x, \phi_u z) < \varepsilon'/3 + \eta_i \text{ for all } u \in [-T'_i, T'_i].$$

Denote $\gamma_i = \alpha_i \alpha^{-1}$ it is clear $\gamma_i: \mathbb{R} \rightarrow \mathbb{R}$ is increasing homeomorphism

with $\gamma_i(0) = 0$ for all i . By continuity of γ_i choose $0 < s_i < \infty$

such that $|u - u'| < s_i$ implies $|\gamma_i(u) - \gamma_i(u')| < r$. Now, take

$$\begin{aligned}
 & d(\phi_{\gamma_{i+1}(u)-\gamma_i(u)} \phi_{\gamma_i(u)}^x, \phi_{\gamma_i(u)}^x) = d(\phi_{\gamma_{i+1}(u)}^x, \phi_{\gamma_i(u)}^x) \\
 & \leq d(\phi_{\gamma_{i+1}(u)}^x, \phi_u^z) + d(\phi_{\gamma_i(u)}^x, \phi_u^z) < 2\varepsilon'/3 + 2\eta_i < \varepsilon'
 \end{aligned}$$

for all $u \in [-T'_i, T'_i]$,

hence $|\gamma_{i+1}(u) - \gamma_i(u)| < r$ because $\varepsilon' < r$ and continuity of γ_{i+1} , γ_i and also because $\gamma_{i+1}(0) = \gamma_i(0) = 0$. From this it is clear that one can take ξ_i with $0 < \xi_i < r$ such that

$u \leq T'_i < u'$ and $|u' - u| < \xi_i$ imply $|\gamma_{i+1}(u') - \gamma_i(u)| < r$.

Fix i and define a sequence $(u_j)_{-\infty}^{\infty}$ of real numbers such that

$u_0 = 0$, $u_j < u_{j+1}$, $u_j \rightarrow \infty$ as $j \rightarrow \infty$, $u_j \rightarrow -\infty$ as $j \rightarrow -\infty$ and

$u_{j+1} - u_j < \min\{\xi_i, s_i\}$ if $u_j \in [0, T'_i]$ for $j \geq 1$ and

$u_{j+1} - u_j < \min\{\xi_i, s_i\}$ if $u_{j+1} \in [-T'_i, 0]$ for $j \leq -1$. It follows

that if we take $t_j = \gamma_i(u_j)$ if $u_j \in [-T'_i, T'_i]$ and $t_j = \gamma_{i+1}(u_j)$

if $u_j \in [-T'_{i+1}, -T'_i] \cup [T'_i, T'_{i+1}]$ and so on that $|t_{j+1} - t_j| < r$

for all $j \in \mathbb{Z}$ and $t_0 = u_0 = 0$ and

$$d(\phi_{t_j} x, \phi_{u_j} z) < \varepsilon' < r \quad \text{for all } j \in \mathbb{Z}.$$

Using theorem 3.2 we have that $x = \phi_t z$ with $|t| < \varepsilon$ and this proves (1).

Now to prove (2) because of (1) we only need to show that A_y is closed in the orbit $(\phi_t z)_{t \in \mathbb{R}}$ with relative topology.

Let z' be a limit point of A_y and assume $z' \in (\phi_t z)_{t \in R}$.

Take $\{z_i\}$ be a sequence in A_y such that $z_i \rightarrow z$. Given $\eta, T > 0$ there are $\alpha_i: R \rightarrow R$ homeomorphisms with $\alpha_i(0) = 0$ such that

$$d(\phi_{\alpha_i(t)} z_i, \psi_t y) < \varepsilon'/6 + \eta/2 \text{ for all } t \in [-T, T], \text{ all } i.$$

Now since all z_i and z are in the same orbit within time distance $\varepsilon < T_0$, there exists an integer N large enough such that

$$d(\phi_t z_i, \phi_t z') < \eta/2 \text{ for all } t \in R, \text{ all } i \geq N.$$

Hence

$$d(\phi_{\alpha_i(t)} z_i, \phi_{\alpha_i(t)} z') < \eta/2 \text{ for all } t \in R, \text{ all } i \geq N.$$

So

$$d(\phi_{\alpha_i(t)} z', \psi_t y) < \varepsilon'/6 + \eta \text{ for all } t \in R.$$

It means that $z' \in A_y$ and this proves (2).

In order to be able to define a function h on Y we need to select a certain point in the set A_y in this way. "A point $x \in A_y$ is called a largest limit of A_y ($\ell.\ell.A_y$) if and only if $x = \phi_w x'$ with $w \geq 0$ for all $x' \in A_y$ ". This point $x = \ell.\ell.A_y$ is the unique point in A_y with this property. Define $h: Y \rightarrow Y$ by

$$h(y) = \ell.\ell.A_y \text{ for all } y \in Y.$$

By uniqueness of $\ell.\ell.A_y$ this function is well defined, from definition of A_y , if $x \in A_y$ we have for every $\eta, T > 0$, there is a homeomorphism $\alpha: R \rightarrow R$ with $\alpha(0) = 0$ s.t.

$$d(\phi_{\alpha(t)}x, \psi_t y) < \varepsilon'/6 + \eta \text{ for all } t \in [-T, T].$$

If we take $\eta < \varepsilon'/2$, then at $t = 0$, $d(x, y) < \varepsilon' < \varepsilon$. And this means that $d(y, \text{l.l.}Ay) < \varepsilon \implies d(h, I) < \varepsilon$ (I is the identity map on Y).

Now to show $h(\text{orbit of } \psi) \subseteq (\text{orbit of } \phi)$. Let $y \in Y$, as we show above the orbit $(\psi_t y)_{t \in \mathbb{R}}$ is $\varepsilon'/6$ -traced by a unique orbit of ϕ say $(\phi_t z)_{t \in \mathbb{R}}$ (i.e., $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ increasing homeomorphism with $\alpha(0) = 0$ such that

$$d(\phi_{\alpha(t)}z, \psi_t y) \leq \varepsilon'/6 \text{ for all } t \in \mathbb{R}.)$$

Let $\psi_u y$ be another point in the $(\psi_t y)_{t \in \mathbb{R}}$. Obviously

$d(\phi_{\alpha(t+u)-\alpha(u)}\phi_{\alpha(u)}z, \psi_t \psi_u y) = d(\phi_{\alpha(t+u)}z, \psi_{t+u} y) \leq \varepsilon'/6$ for all $t \in \mathbb{R}$. Now if we take $\gamma(t) = \alpha(t+u) - \alpha(u)$, $\gamma: \mathbb{R} \rightarrow \mathbb{R}$ is increasing homeomorphism with $\gamma(0) = 0$ (i.e., a reparametrization for the orbit $(\phi_t z)_{t \in \mathbb{R}}$). Therefore $\phi_{\alpha(u)}z \in A_{\psi_u(y)}$ and this means that $h(\text{orbit of } \psi) \subseteq (\text{orbit of } \phi)$.

To show h is continuous, let $\eta, T > 0$, $y \in Y$ and define $A_{y, \eta, T} = \{x: \text{there exists a homeomorphism } \alpha: \mathbb{R} \rightarrow \mathbb{R} \text{ with } \alpha(0) = 0 \text{ such that } d(\phi_{\alpha(t)}x, \psi_t y) < \varepsilon'/6 + \eta \text{ for all } t \in [-T, T]\}$.

Properties of such sets are

- (a) $\eta_1 \geq \eta_2 > 0 \implies A_{y, \eta_1, T} \supseteq A_{y, \eta_2, T}$ for all $y \in Y, T \in \mathbb{R}$,
- (b) $0 < T_1 \leq T_2 \implies A_{y, \eta, T_1} \supseteq A_{y, \eta, T_2}$ for all $y \in Y, \eta \in \mathbb{R}$,
- (c) $0 > \eta_1 \geq \eta_2$ and $0 < T_1 \leq T_2 \implies A_{y, \eta_1, T_1} \supseteq A_{y, \eta_2, T_2}$ for all $y \in Y$,
- (d) if $\{\eta_i\}, \{T_i\}$ are sequences of positive real numbers with $\eta_i \rightarrow 0$ and $T_i \rightarrow \infty$ as $i \rightarrow \infty$, then

$$A_y = \bigcap_i A_{y, \eta_i, T_i}.$$

For more properties we need to prove several lemmas:

Lemma 3.6.

For every $\lambda > 0, y \in Y$ there are $\eta, T > 0$ such that $d(x, A_y) < \lambda$ for all $x \in A_{y, \eta, T}$.

Proof: Assume $\{\eta_i\}, \{T_i\}$ are sequences of positive real numbers with $\eta_i \rightarrow 0, T_i \rightarrow \infty$ as $i \rightarrow \infty$ and also assume $\eta_i < \epsilon'/6$ for all i . Also assume $\{z_i\}$ is a sequence of points such that $z_i \in A_{y, \eta_i, T_i}$ for all i and $z_i \rightarrow z$ as $i \rightarrow \infty$. From definition of A_{y, η_i, T_i} we have that there are $\alpha_i: \mathbb{R} \rightarrow \mathbb{R}$ homeomorphisms with $\alpha_i(0) = 0$ for all i such that

$$* \text{ ---- } d(\phi_{\alpha_i(t)} z_i, \psi_t y) < \epsilon'/6 + \eta_i \text{ for all } t \in [-T_i, T_i], \text{ all } i \in \mathbb{Z}.$$

Since $z_i \rightarrow z$, one can take sequences $\{w_i\}, \{\beta_i\}$ of positive real numbers with $\beta_i \rightarrow 0, w_i \rightarrow \infty$ as $i \rightarrow \infty$ such that

$$d(\phi_{t_i} z_i, \phi_t z) < \beta_i \text{ for } t \in [-w_i, w_i]$$

Also fix N large enough so that for all $j \geq i \geq N$ we have

$$d(\phi_{t_i} z_i, \phi_{t_j} z_j) < \beta_i \text{ for all } t \in [-w_i, w_i]$$

and also without loss of generality take $\beta_i < \epsilon'/6$ for all $i \geq N$. Therefore

$$d(\phi_{\alpha_i(t)} z_i, \phi_{\alpha_i(t)} z) < \beta_i \text{ for all } t \in [\alpha_i^{-1}(-w_i), \alpha_i^{-1}(w_i)].$$

Now to show $\alpha_i^{-1}(w_i) \rightarrow \infty$ as $i \rightarrow \infty$, assume not. (i.e., \exists a positive constant such that $\alpha_i^{-1}(w_i) \leq a$ for all $i \geq N$.) Therefore

$$w_i \leq \alpha_i(a) \text{ for all } i \geq N, \text{ but } w_i \rightarrow \infty \text{ as } i \rightarrow \infty \implies \alpha_i(a) \rightarrow \infty$$

as $i \rightarrow \infty$. Now without loss of generality take N large enough such that a in the domain of α_i for all $i \geq N$. (i.e.,

$a \in [-T_i, T_i]$ for all $i \geq N$.) Using lemma 3.5 and the idea

that $\alpha_j(a) \rightarrow \infty$ as $j \rightarrow \infty$ implies the existence of $t_1 \leq t_2$ are real numbers in $[0, a]$, there exists $j \geq N$ so that t_1 is very close to t_2 to make $d(\psi_{t_1} y, \psi_{t_2} y) < \delta$ and $\alpha_j(t_2) - \alpha_j(t_1) = r$.

The way we defined r in the beginning and using lemma 3.3 we will have

$$d(\phi_{\alpha_j(t_1)} z_j, \phi_{\alpha_j(t_2)} z_j) \geq \gamma > \epsilon' \quad \text{----} **$$

But using * above we have

$$\begin{aligned} d(\phi_{\alpha_j(t_1)} z_j, \phi_{\alpha_j(t_2)} z_j) &\leq d(\phi_{\alpha_j(t_1)} z_j, \psi_{t_1} y) + d(\psi_{t_1} y, \psi_{t_2} y) \\ &+ d(\phi_{\alpha_j(t_2)} z_j, \psi_{t_2} y) < \epsilon'/3 + 2\eta_i + \delta < \epsilon'/3 + \epsilon'/3 + \epsilon'/12 = 9\epsilon'/12 \end{aligned}$$

and this contradicts **. Hence $\alpha_j^{-1}(w_j) \rightarrow \infty$, similarly $\alpha_j^{-1}(w_j) \rightarrow -\infty$ as $j \rightarrow \infty$. So one can assume that there are sequence of positive real numbers $\{\beta_i\}, \{v_i\}$ with $\beta_i \rightarrow 0$, $v_i \rightarrow \infty$ such that

$$d(\phi_{\alpha_i}(t)z_i, \phi_{\alpha_i}(t)z) < \beta_i \text{ for all } t \in [-v_i, v_i].$$

Using * we have

$$\begin{aligned} d(\phi_{\alpha_i}(t)z, \psi_t y) &\leq d(\phi_{\alpha_i}(t)z, \phi_{\alpha_i}(t)z_i) + d(\phi_{\alpha_i}(t)z_i, \psi_t y) \\ &< \epsilon'/6 + \eta_i + \beta_i \text{ for all } t \in [-k_i, k_i] \end{aligned}$$

(where $k_i = \min \{v_i, T_i\}$).

Hence $z \in A_{y, \eta_i + \beta_i, k_i}$ for all i . But $\eta_i + \beta_i \rightarrow 0$, $k_i \rightarrow \infty$ as $i \rightarrow \infty$, therefore $z \in A_y$. Now if we assume that $d(z_i, A_y) \geq \lambda$ for all i , then $d(z, A_y) \geq \lambda$. This is a contradiction and finishes the proof of the lemma. \square

Lemma 3.7.

For every $\lambda > 0$ there are $\eta, T > 0$ such that for every $y \in Y$, $d(x, A_y) < \lambda$ for all $x \in A_{y, \eta, T}$.

Proof: For every point $y \in Y$, let U_y be a neighbourhood of y with the property that

$$d(\psi_t y, \psi_t y') < \frac{\eta_y}{2} \text{ for all } t \in [-T_y, T_y], \text{ all } y' \in U_y,$$

where η_y, T_y are taken as in lemma 3.6 (i.e., $d(x, A_y) < \lambda/2$ for all $x \in A_{y, \eta_y, T_y}$). Let x be any point in $A_{y', \frac{\eta_y}{2}, T_y}$.

There exists $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ homeomorphism with $\alpha(0) = 0$ such that

$$d(\phi_{\alpha(t)}x, \psi_t y') < \varepsilon'/6 + \frac{\eta_y}{2} \text{ for } t \in [-T_y, T_y].$$

Hence

$$d(\phi_{\alpha(t)}x, \psi_t y) < \varepsilon'/6 + \eta_y \text{ for } t \in [-T_y, T_y].$$

Obviously this means that $A_{y', \frac{\eta_y}{2}, T_y} \subseteq A_{y, \eta_y, T_y}$ for all $y' \in U_y$.

By compactness of Y , there are points y_1, y_2, \dots, y_k with an open cover U_1, U_2, \dots, U_k and $\eta_1, \eta_2, \dots, \eta_k, T_1, T_2, \dots, T_k$ taken as above.

Let $\eta = \min_{1 \leq i \leq k} \{\eta_i\}$, $T = \max_{1 \leq i \leq k} \{T_i\}$. Now if y' be any point in Y ,

there exists j , $1 \leq j \leq k$ such that $y' \in U_j$. But $A_{y', \eta/2, T} \subseteq A_{y_j, \eta, T}$,

therefore if we take $z \in A_{y', \eta/2, T}$ we will have $d(z, A_{y_j}) < \lambda/2$.

Since $A_{y'} \subseteq A_{y', \eta/2, T} \subseteq A_{y_j, \eta, T}$, we have

$$d(A_{y'}, A_{y_j}) < \lambda/2. \text{ Hence}$$

$$d(z, A_{y'}) \leq d(z, A_{y_j}) + d(A_{y_j}, A_{y'}) < \lambda. \quad \square$$

Lemma 3.8.

Let $\{y_i\}, \{z_i\}$ be sequences of points in Y . Then if

$z_i \in A_{y_i}$ for all i , $z_i \rightarrow z$, $y_i \rightarrow y$, then $z \in A_y$.

Proof: Take $\{\lambda_i\}$ to be a sequence of positive real number such that $\lambda_i \rightarrow 0$. Using lemma 3.7, there are $\eta_i, T_i > 0$ such that $d(x, A_y) < \lambda_i$ for all $x \in A_{y, \eta_i, T_i}$ and for every $y \in Y$. Since $y_i \rightarrow y$ there is a subsequence $\{y_{j_i}\}$ of $\{y_i\}$ satisfies

$$d(\psi_t y, \psi_t y_{j_i}) < \frac{\eta_i}{2} \text{ for } t \in [-T_i, T_i], \text{ all } i.$$

But $z_{j_i} \in A_{y_{j_i}}$, therefore there are homeomorphisms $\alpha_i: \mathbb{R} \rightarrow \mathbb{R}$ with $\alpha_i(0) = 0$ such that

$$d(\phi_{\alpha_i(t)} z_{j_i}, \psi_t y_{j_i}) < \varepsilon'/6 + \frac{\eta_i}{2} \text{ for } t \in [-T_i, T_i].$$

So

$$d(\phi_{\alpha_i(t)} z_{j_i}, \psi_t y) \leq d(\phi_{\alpha_i(t)} z_{j_i}, \psi_t y_{j_i}) +$$

$$d(\psi_t y_{j_i}, \psi_t y) < \varepsilon'/6 + \eta_i \text{ for } t \in [-T_i, T_i].$$

It means that $z_{j_i} \in A_{y, \eta_i, T_i}$ for all i . By

Lemma 3.7, we have $d(z_{j_i}, A_y) < \lambda_i$ for all i . But $z_{j_i} \rightarrow z$,

$\lambda_i \rightarrow 0$, therefore $d(z, A_y) = 0$. A_y is closed set, hence $z \in A_y$. \square

Now, we are going to use the above lemmas to show that h is continuous, for this assume $\{y_i\}, \{z_i\}$ are sequence of points in Y such that $z_i = \ell.\ell. A_{y_i}$ for all i and assume $y_i \rightarrow y$ and $z = \ell.\ell. A_y$. We want to show that $\{z_i\}$ is a convergent sequence, $z_i \rightarrow z$? From compactness of Y , $\{z_i\}$ has a convergent subsequence, so without loss of generality assume $z_i \rightarrow z'$, obvious from lemma 3.8. $z' \in A_y$.

Let x be any point in A_y and let $\{\lambda_i\}$ be a convergent sequence of positive real numbers with 0 as the only limit. Choose $\eta_i, T_i > 0$ as in lemma 3.7. If $y_i \rightarrow y$, there exists a subsequence $\{y_{k_i}\}$ such that

$$d(\psi_{t y_{k_i}}, \psi_t y) < \frac{\eta_i}{2} \quad \text{for } t \in [-T_i, T_i], \text{ all } i.$$

Since $x \in A_y$, there exists a homeomorphism $\alpha_i: \mathbb{R} \rightarrow \mathbb{R}$ with $\alpha(0) = 0$ such that

$$d(\phi_{\alpha_i(t)} x, \psi_t y) < \varepsilon'/6 + \frac{\eta_i}{2} \quad \text{for } t \in [-T_i, T_i].$$

Therefore

$$d(\phi_{\alpha_i(t)} x, \psi_t y_{k_i}) < \varepsilon'/6 + \eta_i \quad \text{for } t \in [-T_i, T_i], \text{ all } i$$

From this we have $x \in A_{y_{k_i}, \eta_i, T_i}$ and using lemma 3.7 implies

that $d(x, A_{y_{k_i}}) < \lambda_i$ for all i . Choose $x_{k_i} \in A_{y_{k_i}}$ such that

$d(x, x_{k_i}) = d(x, A_{y_{k_i}}) = \lambda_i$. (This can be done because $A_{y_{k_i}}$ is

closed.) Obviously $x_{k_i} \rightarrow x$. Since $z_{k_i} = \text{l.l. } A_{y_{k_i}}$, there are

$w_{k_i} \geq 0$ such that $z_{k_i} = \phi_{w_{k_i}} x_{k_i}$. Also $z_{k_i} \rightarrow z'$, hence $z' = \phi_w x$

with $w \geq 0$ for every $x \in A_y$ and this means that z' is the larger limit of A_y . But the larger limit of A_y is unique, so $z = z'$.

So we proved that every convergent subsequence of $\{z_i\}$ has z as

a limit and this means that $z_i \rightarrow z$. This shows that h is continuous

and complete the proof of theorem 4. \square

Remark 3.1

(a) For the set A_y , $y \in Y$ in the above theorem one can select another point in A_y in the following way:

"A point $x \in A_y$ is called a smallest limit of A_y (s.l. A_y) if and only if $x = \phi_w x'$ with $w \leq 0$ for all $x' \in A_y$."

It is obvious this point $x = \text{s.l.}A_y$ is a unique point in A_y with this property. Define $h': Y \rightarrow Y$ by

$$h'(y) = \text{s.l.}A_y.$$

Using the same idea as the proof of theorem 4 one can show also that h' is a conjugating map near I (i.e., $d(h', I) < \epsilon$, $h'(\text{orbit of } \psi) \subseteq (\text{orbit of } \phi)$.) Hence the above map h in the theorem 4 is not unique.

(b) In theorem 4 if Y is a compact manifold and ϵ sufficiently small, then $d(h, I) < \epsilon$ implies that h maps Y onto Y ([17] p.36).

(c) In theorem 4, let $y \in Y$ and assume y is a periodic point with respect to flow ψ (i.e., $\exists t' > 0$ such that $\psi_{nt'} y = y$ for all n). As we know $(\psi_t y)_{t \in \mathbb{R}}$ is $\epsilon'/6$ -traced by a unique orbit $(\phi_t z)_{t \in \mathbb{R}}$ (i.e., $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ homeomorphism with $\alpha(0) = 0$ such that

$$d(\phi_{\alpha(t)} z, \psi_t y) \leq \epsilon'/6 \text{ for all } t \in \mathbb{R}.)$$

Define $\gamma: \mathbb{R} \rightarrow \mathbb{R}$ by

$$\gamma(t) = \alpha(t+t') - \alpha(t'),$$

obvious that γ is a homeomorphism and $\gamma(0) = 0$.

Therefore

$$d(\phi_{\gamma(t)}\phi_{\alpha(t')}z, \psi_t y) = d(\phi_{\alpha(t+t')}z, \psi_{t+t'}y) \leq \varepsilon'/2 \text{ for all } t \in \mathbb{R}.$$

It means that $\phi_{\alpha(t')}z \in A_y$, since $\text{diam } A_y < \varepsilon$ implies that

$$\phi_{\alpha(t')}z = \phi_w z \text{ with } |w| < \varepsilon. \text{ Hence } z \text{ is also a periodic point.}$$

So this means every periodic orbit of ψ is traced by a periodic orbit ϕ (i.e., h maps periodic orbits of ψ into periodic orbits of ϕ). If y is a fixed point (i.e., $\psi_t y = y$ for all $t \in \mathbb{R}$), then $t' = 0$ and this means that $z = \phi_w z$ with $|w| < \varepsilon$ (i.e., z is of period w). But ε is chosen in the theorem 3 to be $< T_0/2$, this is a contradiction and hence this means that for every flow ψ on Y with $(\phi_t, \psi_t) < \delta$ for all $t \in [0, 1]$ has no fixed points.

(d) In theorem 4 assume the orbit $(\psi_t y)_{t \in \mathbb{R}}$ is $\varepsilon'/6$ -traced by an orbit $(\phi_t z)_{t \in \mathbb{R}}$ (i.e., $h(y) = z$). As we know such an orbit $(\phi_t z)_{t \in \mathbb{R}}$ is unique. Also assume $z' \in (\phi_t z)_{t \in \mathbb{R}}$ (i.e., $\exists t'$ such that $z' = \phi_{t'} z$). Now as we know

$d(\phi_{\alpha(t)}z, \psi_t y) \leq \varepsilon'/6$ for all $t \in \mathbb{R}$ and for some homeomorphisms $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ with $\alpha(0) = 0$, so if we take $\gamma(t) = \alpha(t+t'') - \alpha(t'')$ [γ is a homeomorphism with $\alpha(0) = 0$.], then

$$d(\phi_{\gamma(t)}\phi_{\alpha(t'')}z, \psi_{t+t''}y) = d(\phi_{\alpha(t+t'')}z, \psi_{t+t''}y) \leq \varepsilon'/6$$

for all $t \in \mathbb{R}$, where $t'' = \alpha^{-1}(t')$.

This means that $\phi_{t'} z \in A_{\psi_{t''}} y \implies z' \in A_{\psi_{t''}} y$. Hence this shows that for every $z' \in (\phi_t z)_{t \in R}$ there exists $y' \in (\psi_t y)_{t \in R}$ such that $z' \in A_{y'}$. (i.e., $y' = \psi_{t''} y$)

Lemma 3.9.

Let $\{\alpha_j\}$ be a family of continuous increasing functions from $[-w_j, w_j]$ into R with $\alpha_j(0) = 0$ for all j and assume $w_j \rightarrow \infty$ as $j \rightarrow \infty$ and assume $\alpha_j(w_j) \leq b$ for all j (b is a constant positive real number). Then for every $\lambda, \beta > 0$, there exists j and there are t_1, t_2 in $[0, w_j]$ with $t_1 < t_2$ such that $t_2 - t_1 = \beta$, $\alpha_j(t_2) - \alpha_j(t_1) < \lambda$.

Proof: For each j , let $\{s_{j_k}^i\}_{k=0}^{n_j}$ be the partition of $[0, w_j]$

(i.e., $s_{j_0} = 0 < s_{j_1} < \dots < s_{j_{n_j}} = w_j$) with $s_{j_{k+1}} - s_{j_k} = \beta$

for all $0 \leq k \leq n_j - 1$. It is clear that $n_j \rightarrow \infty$ as $j \rightarrow \infty$. Now assume $\alpha_j(s_{j_{k+1}}) - \alpha_j(s_{j_k}) \geq \lambda$ for all j and $0 \leq k \leq n_j - 1$. It means that $\alpha_j(w_j) - \alpha_j(0) \geq \lambda \cdot n_j \rightarrow \infty$ as $j \rightarrow \infty$. This is a contradiction with the hypothesis. \square

Corollary

If $\{\alpha_j\}$ as in lemma 3.8 and if $b < \alpha_j(-w_j)$ for all j (b is a negative constant real number), then for every $\lambda, \beta > 0$, there exists j and there are t_1, t_2 in $[-w_j, 0]$ with $t_1 < t_2$ such that $t_2 - t_1 = \beta$, $\alpha_j(t_2) - \alpha_j(t_1) < \lambda$.

Proof: Exactly similar as above.

Theorem 5.

In the statement of theorem 4 if ψ is also expansive, then h is injective.

In particular, for every $y \in Y$ with $h(y) = z$,

$$h / (\psi_t y)_{t \in R} : (\psi_t y)_{t \in R} \rightarrow (\phi_t z)_{t \in R}$$

is a homeomorphism.

Proof: From remark 3.1(c) we know ψ has no fixed points, so take ϵ in the theorem 4 to be also $< T'_0 / 2$ (T'_0 is taken from lemma 3.3 with respect to ψ), take $\xi > 0$ such that $d(\psi_\epsilon x, x) \geq \xi$ for all $x \in Y$. Now using the expansiveness condition of ψ choose e such that $d(\psi_{s(t)} y_1, \psi_t y_2) < 4e$ for all $t \in R$ implies that $y_1 = \phi_w y_2$ with $|w| < \epsilon$ for all $y_1, y_2 \in Y$ and $s: R \rightarrow R$ continuous with $s(0) = 0$. In theorem 4 choose r small enough such that $x = \psi_r y \implies d(x, y) < e$ for all $x, y \in Y$. Also in theorem 4, take ϵ' with $\epsilon' < \xi$, $\epsilon' < e$. Now define for every $z \in Y$ in which $h(y) = z$ a set

$$B_z = \left\{ y: y \in Y \text{ such that } z \in A_y \right\}$$

Fix $y \in B_z$ we want to show first that $B_z \subseteq (\psi_t y)_{t \in R}$ and

$\text{diam } B_z < \varepsilon$. From definition A_y if we have two sequences $\{\eta_i\}$, $\{T_i\}$ of positive real number with $\eta_i \rightarrow 0$, $T_i \rightarrow \infty$ as $i \rightarrow \infty$, then there are a family $\{\alpha_i\}$ of continuous increasing functions from $[-T_i, T_i]$ into R with $\alpha_i(0) = 0$ for all i such that

$$* \quad d(\phi_{\alpha_i(t)}^z, \psi_t y) < \varepsilon'/6 + \eta_i \text{ for all } t \in [-T_i, T_i], \text{ all } i.$$

Now without loss of generality assume $\eta_i < \varepsilon'/6$ for all i . For $j \geq i$ we have

$$\begin{aligned} d(\phi_{\alpha_j(t)}^z, \phi_{\alpha_i(t)}^z) &\leq d(\phi_{\alpha_j(t)}^z, \psi_t y) + d(\phi_{\alpha_i(t)}^z, \psi_t y) \\ &< \varepsilon'/3 + 2\eta_i \text{ for all } t \in [-T_i, T_i]. \end{aligned}$$

Therefore for all $j \geq i$ we have

$$d(\phi_{\alpha_j(t)}^z, \phi_{\alpha_i(t)}^z) < \varepsilon' < \gamma \text{ all } t \in [-T_i, T_i]$$

Using lemma 3.3 and continuity of α_i , α_j with $\alpha_i(0) = \alpha_j(0) = 0$ we should have,

$$* \quad |\alpha_j(t) - \alpha_i(t)| < r \text{ for all } j \geq i, \text{ all } t \in [-T_i, T_i].$$

Now to show $\alpha_i(T_i) \rightarrow \infty$ as $i \rightarrow \infty$, assume not (i.e., $\exists b$, b is positive constant s.t. $\alpha_i(T_i) \leq b$), using lemma 3.9, $(\exists j)$ and there are $t_1 < t_2$ in $[0, T_j]$ such that $t_2 - t_1 = \varepsilon$ but $\alpha_j(t_2) - \alpha_j(t_1)$ is small enough so that $d(\phi_{\alpha_j(t_2)}^z, \phi_{\alpha_j(t_1)}^z) < \delta$. But

$$d(\phi_{\alpha_j(t_2)}^z, \psi_{t_2} y) < \varepsilon'/6 + \eta_j,$$

$$d(\phi_{\alpha_j(t_1)}^z, \psi_{t_1} y) < \varepsilon'/6 + \eta_j.$$

Therefore

$$d(\psi_{t_1} y, \psi_{t_2} y) \leq d(\phi_{\alpha_j(t_1)}^z, \psi_{t_1} y) + d(\phi_{\alpha_j(t_1)}^z, \phi_{\alpha_j(t_2)}^z)$$

$$+ d(\phi_{\alpha_j(t_2)}^z, \psi_{t_2} y) < \epsilon'/3 + 2\eta_j + \delta < \epsilon'/3 + \epsilon'/3 + \epsilon'/12 < \epsilon',$$

but since $t_2 - t_1 = \epsilon \implies d(\psi_{t_1} y, \psi_{t_2} y) \geq \xi > \epsilon'$.

This is a contradiction. Therefore $\alpha_j(T_j) \rightarrow \infty$. So without loss of generality we can assume that that we have $\{\alpha_j\}$ sequence of increasing maps from $[-T_i, T_i]$ into R such that for fixed i we have

$$(a) \quad |\alpha_{j+1}(t) - \alpha_j(t)| < r \quad \text{for all } t \in [-T_j, T_j], j \geq i$$

(b) range of α_j is contained in the interior of the range of α_{j+1} .

Now define $\alpha: R \rightarrow R$ be a homeomorphism as follows:

$$\alpha = \alpha_i \text{ on } [-T_i, T_i],$$

as we know $|\alpha_{i+1}(T_i) - \alpha_i(T_i)| < r$ so there is a continuous decreasing function on $[T_i, T_{i+1}]$ such that

$$\alpha(T_i) = \alpha_i(T_i), \quad \alpha(T_{i+1}) = \alpha_{i+1}(T_{i+1}) \text{ and}$$

$$|\alpha(t) - \alpha_{i+1}(t)| < r \quad \text{for all } t \in [T_i, T_{i+1}]$$

if we carry on in the same manner we have such a homeomorphism

$\alpha: R \rightarrow R$ with $\alpha(0) = 0$. Now pick $t \in R$, say first $t > 0$,

$t \in [T_j, T_{j+1}]$ implies that

$$d(\phi_{\alpha(t)}^z, \phi_{\alpha_{j+1}(t)}^z) < e,$$

so

$$\begin{aligned} d(\phi_{\alpha(t)}^z, \psi_t y) &\leq d(\phi_{\alpha(t)}^z, \phi_{\alpha_{j+1}(t)}^z) + \\ d(\phi_{\alpha_{j+1}(t)}^z, \psi_t y) &< e + \varepsilon'/6 + \eta_j \\ &< 2e. \end{aligned}$$

So this means that if $z \in A_y$, $\exists \alpha: \mathbb{R} \rightarrow \mathbb{R}$ homeomorphism with $\alpha(0) = 0$ such that $d(\phi_{\alpha(t)}^z, \psi_t y) < 2e$ all $t \in \mathbb{R}$.

Now assume $y_1, y_2 \in B_z \implies z \in A_{y_1}, z \in A_{y_2}$ using the above conclusion implies that there are $\alpha_1, \alpha_2: \mathbb{R} \rightarrow \mathbb{R}$ homeomorphisms with $\alpha_1(0) = \alpha_2(0) = 0$ respectively such that

$$\begin{aligned} d(\phi_{\alpha_1(t)}^z, \psi_t y_1) &< 2e \quad \text{all } t \in \mathbb{R}, \\ d(\phi_{\alpha_2(t)}^z, \psi_t y_2) &< 2e \quad \text{all } t \in \mathbb{R}. \end{aligned}$$

So $d(\psi_{\alpha_1^{-1}(t)}^{-1} y_1, \psi_{\alpha_2^{-1}(t)}^{-1} y_2) < 4e$ all $t \in \mathbb{R}$, using expansive

definition of ψ we have $y_1 = \phi_w y_2$ with $|w| < \varepsilon$, and this means that $B_z \subseteq (\psi_t y)_{t \in \mathbb{R}}$ and time diameter of $B_z < \varepsilon$.

Now to show B_z is closed, assume $y_i \rightarrow x$ where $y_i \in B_z$ all i , so $z \in A_{y_i}$ all i . lemma 3.8 implies $z \in A_x \implies x \in B_z$, therefore B_z is closed set.

Now to prove that y is the smallest limit of B_z if and only if z is the largest limit of A_y . Assume $y = \text{s.l. } B_z$ but z is not l.l. A_y , so there is $x \in A_y$ such that $x = \phi_w z$ with $w > 0$ (definition of l.l. A_y). Given $\eta, T > 0$ there is $\gamma: \mathbb{R} \rightarrow \mathbb{R}$ homeomorphism with $\gamma(0) = 0$ such that

$$d(\phi_{\gamma(t)}x, \psi_t y) < \varepsilon'/6 + \eta \text{ all } t \in [-T, T].$$

Let v be a negative real number such that $\gamma(v) = -w$, define $\lambda: \mathbb{R} \rightarrow \mathbb{R}$ to be $\lambda(t) = \gamma(t+v) - \gamma(v)$, it is clear γ is a homeomorphism with $\gamma(0) = 0$,

$$d(\phi_{\lambda(t)}z, \psi_t \psi_v y) < \varepsilon'/6 + \eta \text{ all } t \in [-T, T]$$

Hence $z \in A_{\psi_v y} \implies \psi_v y \in B_z$. This is a contradiction because $v < 0$ and $y = \text{s.l.} B_z$. The other way is just similar so we have

$$z = \text{l.l.} A_y \text{ if and only if } y = \text{s.l.} B_z.$$

Hence h should be 1-1 because if $h(y_1) = h(y_2) = z$, then y_1, y_2 are both smallest limit of B_z . This means that $y_1 = y_2$.

Also if $h(y) = z$ one can define on the orbit $(\phi_t z)_{t \in \mathbb{R}}$ a

$$\text{map } p: (\phi_t z)_{t \in \mathbb{R}} \rightarrow (\psi_t y)_{t \in \mathbb{R}} \text{ by}$$

$$p(z) = \text{s.l.} B_z$$

using similar ideas to the proof of theorem 4 one can easily show that p is continuous on $(\phi_t z)_{t \in \mathbb{R}}$ and $p \circ h / (\psi_t y)_{t \in \mathbb{R}} =$

$$h / (\psi_t y)_{t \in \mathbb{R}} \circ p = I \text{ where } I \text{ is the identity map.}$$

Hence $h/(\psi_t y)_{t \in R} : (\psi_t y)_{t \in R} \rightarrow (\phi_t z)_{t \in R}$ is a homeomorphism

and this finishes the proof of theorem 5. \square

Corollary.

If h in the theorem 5 is onto, then h is a homeomorphism.

So in the statement of theorem 4 if Y is a compact manifold and ε sufficiently small and ψ is also expansive, then the stable map h is a homeomorphism on Y .

CHAPTER IV

In this chapter we are going to define a "finite P.O.T.P." for the flows. This notion is important not only for theoretical reasons, but also because it is easier to check finite P.O.T.P. . First we are going to show how the "finite P.O.T.P" will imply P.O.T.P for almost all cases and secondly, we show every topologically stable smooth flow without fixed points on a compact manifold has P.O.T.P.

Let ϕ be a continuous flow on a compact metric space (Y,d) .

Definition 4.1.

A flow ϕ is said to have a "finite pseudo orbit tracing property" (finite P.O.T.P.) if for every $\epsilon > 0$, there is $\delta > 0$ such that every finite $(\delta,1)$ -pseudo orbit is ϵ -traced by an orbit of ϕ .

See chapter II for definition of finite $(\delta,1)$ -pseudo orbit. This definition does not depend on the choice of metric on Y .

Lemma 4.1.

Let ϕ have no fixed points. Then there exists $T_0 > 0$ such that if $0 < T < T_0$, there is $\lambda > 0$ such that $d(x,y) < \lambda$ implies that $d(\phi_T x, y) > \lambda$ for all $x, y \in Y$.

Proof: From lemma 3.3 such T_0 exists with the property that if $0 < T < T_0$, there is $\gamma > 0$ such that $d(\phi_T x, x) \geq \gamma$ all $x \in Y$. Take $\lambda < \gamma/2$ and assume $d(x, y) < \lambda$.

$$d(x, \phi_T x) \leq d(x, y) + d(\phi_T x, y)$$

Hence,

$$d(\phi_T x, y) \geq d(x, \phi_T x) - d(x, y) \geq \gamma - \gamma/2 > \lambda. \quad \square$$

Proposition 4.1.

Every flow with no fixed points and *with* finite P.O.T.P., has P.O.T.P.

Proof: Using lemma 3.3 and without loss of generality assume $0 < \varepsilon < T_0$, $\varepsilon < \frac{1}{2}$ and take γ as in lemma 3.3, choose $0 < \varepsilon' < \gamma/4$, $\varepsilon' < \varepsilon/4$ such that $x = \phi_t y$ with $|t| < \varepsilon'$ implies that $d(x, y) < \varepsilon/4$. Also take $0 < \lambda < \varepsilon'$ which satisfies lemma 4.1 (i.e., $d(x, y) < \lambda \implies d(\phi_{\varepsilon} x, y) > \lambda$) with respect to ε' and take $0 < \delta < \frac{\lambda}{4}$ which satisfies the following:

- (a) Every finite $(\delta, 1)$ -pseudo orbit is $\frac{\lambda}{4}$ -traced by an orbit of ϕ .
- (b) $d(x, y) < \delta \implies d(\phi_t x, \phi_t y) < \frac{\lambda}{4}$ for all $t \in [0, 2]$.
- (c) $x = \phi_t y$ with $|t| < \delta \implies d(x, y) < \frac{\lambda}{4}$.

Let $(\{x_i\}_{-\infty}^{\infty}, \{t_i\}_{-\infty}^{\infty})$ be a $(\delta, 1)$ -pseudo orbit for ϕ with $1 \leq t_i \leq 2$.

From the hypothesis for every integer $k > 0$, there exists

$z_k \in Y$ and a continuous increasing map $\alpha_k: [-\sum_{i=0}^{k-1} t_i, \sum_{i=0}^{k-1} t_i] \rightarrow R$

with $\alpha_k(0) = 0$ such that for $t \geq 0$

$$(*) \quad d(\phi_{\alpha_k(t)} z_k, \phi_{t - \sum_{i=0}^{n-1} t_i} x_n) < \frac{\lambda}{4} \text{ for } \sum_{i=0}^{n-1} t_i \leq t < \sum_{i=0}^n t_i, \quad 0 \leq n \leq k-1,$$

for $t < 0$

$$d(\phi_{\alpha_k(t)} z_k, \phi_{t + \sum_{i=0}^{n-1} t_i} x_{-n}) < \frac{\lambda}{4} \text{ for } -\sum_{i=0}^{n-1} t_i \leq t < -\sum_{i=0}^{n-1} t_i$$

Since Y is compact, so without loss of generality assume

$z_k \rightarrow z$, also call each $[-\sum_{i=0}^{k-1} t_i, \sum_{i=0}^{k-1} t_i]$ domain of α_k . Let $\{T_i\}$

be a sequence of positive real number such that $T_i \rightarrow \infty$ as $i \rightarrow \infty$.

Also without loss of generality we can assume that

$$d(\phi_{T_i} z_i, \phi_{T_i} z) < \frac{\lambda}{4} \text{ for } t \in [-T_i, T_i], \text{ all } i.$$

Hence

$$d(\phi_{\alpha_i(t)} z, \phi_{\alpha_i(t)} z) < \frac{\lambda}{4} \text{ for } t \in [\alpha_i^{-1}(-T_i), \alpha_i^{-1}(T_i)], \text{ all } i.$$

Now we want to show $\alpha_i^{-1}(T_i) \rightarrow \infty$, $\alpha_i^{-1}(-T_i) \rightarrow -\infty$ as $i \rightarrow \infty$. Assume

not, that is, there exists a positive constant real number such

that $\alpha_i^{-1}(T_i) \leq a$ for all i . Since $T_i \rightarrow \infty$, there is a positive

integer N such that a in the domain of α_i ($a \in [-T_i, T_i]$) for all

$i \geq N$). Therefore $T_i \leq \alpha_i(a) \rightarrow \infty$ as $i \rightarrow \infty$, using lemma 3.5 there

exists $j \geq N$ and there are u_1, u_2 in the domain of α_j with $u_1 \leq u_2$

such that $u_2 - u_1 < \delta$ and $\alpha_j(u_2) - \alpha_j(u_1) = \varepsilon$

Now if $\sum_{i=0}^{n-1} t_i \leq u_1 < u_2 < \sum_{i=0}^n t_i$, then

$$d(\phi_{u_1 - \sum_{i=0}^{n-1} t_i}^{n-1} x_n, \phi_{u_2 - \sum_{i=0}^n t_i}^{n-1} x_n) < \lambda/4.$$

Using (*) above we will have

$$\begin{aligned} & d(\phi_{\alpha_j(u_1)}^{z_j}, \phi_{\alpha_j(u_2)}^{z_j}) \leq d(\phi_{\alpha_j(u_1)}^{z_j}, \phi_{u_1 - \sum_{i=0}^{n-1} t_i}^{n-1} x_n) \\ & + d(\phi_{u_1 - \sum_{i=0}^{n-1} t_i}^{n-1} x_n, \phi_{u_2 - \sum_{i=0}^n t_i}^{n-1} x_n) + d(\phi_{\alpha_j(u_2)}^{z_j}, \phi_{u_2 - \sum_{i=0}^n t_i}^{n-1} x_n) \\ & < \frac{\lambda}{4} + \frac{\lambda}{4} + \frac{\lambda}{4} = \frac{3\lambda}{4}. \end{aligned}$$

But $\alpha_j(u_2) - \alpha_j(u_1) = \varepsilon$, using lemma 3.3,

$$d(\phi_{\alpha_j(u_2)}^{z_j}, \phi_{\alpha_j(u_1)}^{z_j}) \geq \gamma > 4\lambda.$$

This is a contradiction. For the other case, that is

$$\sum_{i=0}^{n-1} t_i < u_1 < \sum_{i=0}^n t_i \leq u_2 \text{ we have}$$

$$\begin{aligned} & d(\phi_{u_1 - \sum_{i=0}^{n-1} t_i}^{n-1} x_n, \phi_{u_2 - \sum_{i=0}^n t_i}^n x_{n+1}) \leq d(\phi_{u_1 - \sum_{i=0}^{n-1} t_i}^{n-1} x_n, \phi_{u_2 - \sum_{i=0}^n t_i}^{n-1} x_n) \\ & + d(\phi_{u_2 - \sum_{i=0}^n t_i}^{n-1} x_n, \phi_{u_2 - \sum_{i=0}^n t_i}^n x_{n+1}). \end{aligned}$$

Since $0 \leq u_2 - \sum_{i=0}^n t_i < t_{n+1} < 2$ and $d(\phi_{t_n} x_n, x_{n+1}) < \delta$, we have that

$$d(\phi_{u_2 - \sum_{i=0}^n t_i} \phi_{t_n} x_n, \phi_{u_2 - \sum_{i=0}^n t_i} x_{n+1}) < \lambda/4.$$

Therefore

$$d(\phi_{u_2 - \sum_{i=0}^{n-1} t_i} x_n, \phi_{u_2 - \sum_{i=0}^n t_i} x_{n+1}) < \lambda/4.$$

Using the above inequality, we have

$$d(\phi_{u_1 - \sum_{i=0}^{n-1} t_i} x_n, \phi_{u_2 - \sum_{i=0}^n t_i} x_{n+1}) < \lambda/2.$$

Also (*) implies that

$$d(\phi_{\alpha_j(u_1)} z_j, \phi_{\alpha_j(u_2)} z_j) \leq d(\phi_{\alpha_j(u_1)} z_j, \phi_{u_1 - \sum_{i=0}^{n-1} t_i} x_n) +$$

$$d(\phi_{u_1 - \sum_{i=0}^{n-1} t_i} x_n, \phi_{u_2 - \sum_{i=0}^n t_i} x_{n+1}) + d(\phi_{\alpha_j(u_2)} z_j, \phi_{u_2 - \sum_{i=0}^n t_i} x_{n+1}) < \lambda.$$

Also this is a contradiction because $\alpha_j(u_2) - \alpha_j(u_1) = \varepsilon \implies$

$$d(\phi_{\alpha_j(u_1)} z_j, \phi_{\alpha_j(u_2)} z_j) > \lambda.$$

The same idea as above can be applied when $t < 0$.

Hence without loss of generality one can assume that there exists a positive integer N and a sequence $\{w_i\}$ of positive real numbers with $w_i \rightarrow \infty$ as $i \rightarrow \infty$ such that

$$d[\phi_{\alpha_i(t)} z_i, \phi_{\alpha_i(t)} z_j] < \frac{\lambda}{4} \text{ for } t \in [-w_i, w_i], j \geq i \geq N.$$

Now to show also $\alpha_j(w_j) \rightarrow \infty$, $\alpha_j(-w_j) \rightarrow -\infty$ as $j \rightarrow \infty$. Assume not (i.e., $\exists b > 0$ such that $\alpha_j(w_j) < b$ for all $j \geq N$), using lemma 3.9, there exists $j \geq N$ and there are $u_1, u_2 \in [0, w_j]$ with $u_1 < u_2$ s.t. $u_2 - u_1 = \varepsilon$, $\alpha_j(u_2) - \alpha_j(u_1) < \delta$. Now

$\alpha_j(u_2) - \alpha_j(u_1) < \delta$ implies that $d(\phi_{\alpha_j(u_1)}^{z_j}, \phi_{\alpha_j(u_2)}^{z_j}) < \lambda/4$.

But (*) implies that in the case $\sum_{i=0}^{n-1} t_i \leq u_1 < u_2 < \sum_{i=0}^n t_i$

$$d(\phi_{u_1 - \sum_{i=0}^{n-1} t_i}^{n-1} x_n, \phi_{u_2 - \sum_{i=0}^{n-1} t_i}^{n-1} x_n) \leq d(\phi_{\alpha_j(u_1)}^{z_j}, \phi_{u_1 - \sum_{i=0}^{n-1} t_i}^{n-1} x_n) + d(\phi_{\alpha_j(u_1)}^{z_j}, \phi_{\alpha_j(u_2)}^{z_j}) + d(\phi_{\alpha_j(u_2)}^{z_j}, \phi_{u_2 - \sum_{i=0}^{n-1} t_i}^{n-1} x_n) < \frac{3\lambda}{4}.$$

This is a contradiction because $u_2 - u_1 = \varepsilon$. lemma 3.3

$$\Rightarrow d(\phi_{u_1 - \sum_{i=0}^{n-1} t_i}^{n-1} x_n, \phi_{u_2 - \sum_{i=0}^{n-1} t_i}^{n-1} x_n) \geq \gamma > \lambda/4.$$

Now if $\sum_{i=0}^{n-1} t_i < u_1 < \sum_{i=0}^n t_i \leq u_2$, then

using the fact that $\alpha_j(u_2) - \alpha_j(u_1) < \delta$, we have

$$d(\phi_{u_1 - \sum_{i=0}^{n-1} t_i}^{n-1} x_n, \phi_{u_2 - \sum_{i=0}^n t_i}^n x_{n+1}) \leq d(\phi_{\alpha_j(u_1)}^{z_j}, \phi_{u_1 - \sum_{i=0}^{n-1} t_i}^{n-1} x_n) + d(\phi_{\alpha_j(u_1)}^{z_j}, \phi_{\alpha_j(u_2)}^{z_j}) + d(\phi_{\alpha_j(u_2)}^{z_j}, \phi_{u_2 - \sum_{i=0}^n t_i}^n x_{n+1}) < \frac{3\lambda}{4}.$$

Now since $d(\phi_{t_n} x_n, x_{n+1}) < \delta$. $0 < u_2 - \sum_{i=0}^n t_i < 2$ implies that

$$d(\phi_{u_1 - \sum_{i=0}^{n-1} t_i} x_n, \phi_{u_2 - \sum_{i=0}^{n-1} t_i} x_n) \leq d(\phi_{u_1 - \sum_{i=0}^{n-1} t_i} x_n, \phi_{u_2 - \sum_{i=0}^n t_i} x_{n+1})$$

$$+ d(\phi_{u_2 - \sum_{i=0}^n t_i} \phi_{t_n} x_n, \phi_{u_2 - \sum_{i=0}^n t_i} x_{n+1}) < \frac{3\lambda}{4} + \frac{\lambda}{4} = \lambda.$$

This also is a contradiction because $u_2 - u_1 = \lambda$, lemma 3.3

$$\Rightarrow d(\phi_{u_1 - \sum_{i=0}^{n-1} t_i} x_n, \phi_{u_2 - \sum_{i=0}^{n-1} t_i} x_n) \geq \gamma > \lambda/4.$$

Same idea as above can be applied when $t < 0$.

Hence without loss of generality one can assume that there exists a positive integer N such that each α_j , $j \geq N$ has a domain $[-w_j, w_j]$ such that

- (i) $w_j \rightarrow \infty$ as $j \rightarrow \infty$,
- (ii) $d(\phi_{\alpha_i(t)} z_i, \phi_{\alpha_i(t)} z_j) < \frac{\lambda}{4}$ for all $t \in [-w_j, w_j]$,
all $j \geq i \geq N$,
- (iii) the range of α_i is contained in the interior of the range of α_{i+1} , for all $i \geq N$.

Fix $i \geq N$ and decompose R as union of the following closed intervals $\dots \cup [-w_{i+2}, -w_{i+1}] \cup [-w_{i+1}, -w_i] \cup [-w_i, w_i]$
 $\cup [w_i, w_{i+1}] \cup [w_{i+1}, w_{i+2}] \cup \dots$

From (*) we have

$$\begin{aligned}
 & d(\phi_{\alpha_{j+1}}(t)^{z_{j+1}}, \phi_{\alpha_j}(t)^{z_j}) \\
 & \leq d(\phi_{\alpha_{j+1}}(t)^{z_{j+1}}, \phi_{t - \sum_{i=0}^{n-1} t_i}^{x_n}) + d(\phi_{\alpha_j}(t)^{z_j}, \phi_{t - \sum_{i=0}^{n-1} t_i}^{x_n}) \\
 & < \frac{\lambda}{4} + \frac{\lambda}{4} = \lambda/2 \text{ for all } t \in [-w_j, w_j], \text{ all } j \geq i \text{ also from}
 \end{aligned}$$

$$(ii) \quad d(\phi_{\alpha_j}(t)^{z_j}, \phi_{\alpha_j}(t)^{z_{j+1}}) < \lambda/4 \text{ for all } t \in [-w_j, w_j],$$

apply lemma 4.1 implies that

$$|\alpha_{j+1}(t) - \alpha_j(t)| < \varepsilon' \text{ for all } t \in [-w_j, w_j], \text{ all } j \geq i$$

Define $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ to be $\alpha = \alpha_i$ on $[-w_i, w_i]$, now as we know

$$|\alpha_i(w_i) - \alpha_{i+1}(w_i)| < \varepsilon', \text{ so there is a continuous decreasing}$$

function on $[w_i, w_{i+1}]$ such that $\alpha(w_i) = \alpha_i(w_i), \alpha(w_{i+1}) = \alpha_{i+1}(w_{i+1})$

and $|\alpha(t) - \alpha_{i+1}(t)| < \varepsilon'$ for $t \in [w_i, w_{i+1}]$, if we carry on in

the same manner we will have such a homeomorphism $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ with

$\alpha(0) = 0$. Now pick $t \in \mathbb{R}$, say first $t > 0$ and without loss of

generality assume $t \in [w_j, w_{j+1}]$. So $|\alpha(t) - \alpha_{j+1}(t)| < \varepsilon'$,

using (*) and (iii) and assuming $\sum_{i=0}^{n-1} t_i \leq t < \sum_{i=0}^n t_i$ we will have

$$d(\phi_{\alpha(t)}^z, \phi_{t - \sum_{i=0}^{n-1} t_i}^{x_n}) \leq d(\phi_{\alpha(t)}^z, \phi_{\alpha_{j+1}}(t)^z)$$

$$+ d(\phi_{\alpha_{j+1}}(t)^z, \phi_{\alpha_{j+1}}(t)^{z_{j+1}}) + d(\phi_{\alpha_{j+1}}^{z_{j+1}}, \phi_{t - \sum_{i=0}^{n-1} t_i}^{x_n})$$

$$< \varepsilon/4 + \frac{\lambda}{4} + \frac{\lambda}{4} < \varepsilon.$$

Similar ideas can be applied when $t < 0$. Using proposition 2.1 ϕ has P.O.T.P. and the proof is complete. \square

Proposition 4.2.

Let ϕ be a continuous flow on a compact metric space with the property: For every $\varepsilon > 0$, there is $\delta > 0$ such that every finite $(\delta, 1)$ -pseudo orbit $(\{x_i\}_{-k}^k, \{t_i\}_{-k}^k)$ with $1 \leq \lambda_i \leq 2$, is ε -traced by an orbit of ϕ . Then ϕ has the finite pseudo orbit tracing property.

Proof: The proof is similar to the one in proposition 2.1.

Corollary.

In the statement of proposition 4.2 if ϕ has no fixed points, then ϕ has P.O.T.P.

Let ϕ be a C^1 flow on a compact manifold M generated by a vector field $X = \phi'$, ([16]) and let $\mathcal{L}(\phi) = \{x; \phi'x = 0\}$ (i.e., $\mathcal{L}(\phi)$ is a set of fixed points).

Now given a non-zero vector $Y \in T_x M$ where $x \notin \mathcal{L}(\phi)$ and define the inclination of Y relative to ϕ to be the length of the normalized difference, that is

$$\sigma(Y) = \left\| \frac{1}{\|Y\|} Y - \frac{1}{\|\phi'_x\|} \phi'_x \right\|$$

Lemma 4.2. (Cf. [18] lemma 9)

Given $\varepsilon > 0$ and a flow ϕ on M . Suppose γ is a C^1 curve in M [an embedded closed interval or circle] such that at each point x in the image of γ one of the following conditions hold:

- (i) $\|\dot{\phi}_x\| < \varepsilon/2$, or
- (ii) $x \notin \mathcal{L}(\phi)$, and γ has inclination $\sigma < \varepsilon/\|\dot{\phi}\|$ at x .

Then, given any neighbourhood U of the image of γ , there exists a flow ψ on M satisfying

- (a) $\dot{\psi} = \dot{\phi}$ off U
- (b) $\|\dot{\psi} - \dot{\phi}\| < \varepsilon$ on M .
- (c) γ is a (segment of an) integral curve of ψ .

Now using the corollary of proposition 4.2 and the above lemmas one can show the following important result.

Theorem 6.

Let ϕ be a C^1 flow without fixed points on a compact manifold M . If ϕ is topologically stable, then ϕ has pseudo orbit tracing property.

Proof: Given $\varepsilon > 0$, without loss of generality take T_0 as in lemma 4.1 and assume $0 < \varepsilon < T_0$, choose $0 < \varepsilon' < \varepsilon/3$ such that

if $x = \phi_t y$ with $|t| < \varepsilon'$ implies that $d(x, y) < \varepsilon/3$. Take $\varepsilon'' < \varepsilon'$ as in lemma 4.1 (i.e., $d(x, y) < \varepsilon'' \implies d(\phi_{\varepsilon'}, x, y) > \varepsilon''$).

Also take $0 < \xi < \varepsilon''$ such that $d(x, y) < \xi$ implies that $d(\phi_t x, \phi_t y) < \varepsilon''$ for all $t \in [0, 2]$. Using definition 3.2 (i.e., definition of topologically stable), there exists $\delta < \varepsilon/2$ such that δ satisfying the definition 3.2 with respect to $\xi/2$. Take $0 < \delta' < \delta$ such that for every C^1 -flow η on M

$\|\eta - \phi\| < \delta \implies d(\eta_t, \phi_t) < \delta$ for $t \in [0, 2]$. Now let

$(\{x_i\}_0^k, \{t_i\}_0^k)$ be a pair of sequences with $1 \leq t_i \leq 2$ all $0 \leq i \leq k$,

where $x_i \in M$ all $0 \leq i \leq k$. Later on we are going to fix the value of λ , but now choose a sequence of distinct points $\{x'_i\}_0^k$ in M with the property

$$d(\phi_t x'_i, \phi_t x_i) < \frac{\lambda}{3} \text{ for all } t \in [0, 2].$$

This can easily be done and without loss of generality we are assuming here that all t_i , $0 \leq i \leq k$ are positive (for negative values the same interpretation can be carried on). So

$$\begin{aligned} d(\phi_{t_i} x'_i, x'_{i+1}) &\leq d(\phi_{t_i} x'_i, \phi_{t_i} x_i) + d(\phi_{t_i} x_i, x_{i+1}) \\ &\quad + d(x_{i+1}, x'_{i+1}) < \lambda/3 + \lambda/3 + \lambda/3 = \lambda, \\ &\text{for } 0 \leq i \leq k-1. \end{aligned}$$

Now take $0 < \lambda < \delta'$, λ is small enough such that one can take a C^1 curve $\gamma: [0, \sum_{i=0}^{k-1} t_i] \rightarrow M$ with the following properties:

(a) γ is a closed curve in M .

(b) $\gamma(\sum_{i=0}^{n-1} t_i) = x_n, 0 \leq n \leq k$

(c) γ has inclination $< \frac{\delta'}{\|\dot{\phi}\|}$ at every point x in

the image of γ .

Using lemma 4.2, there exists a C^1 flow ψ on M such that

(a') γ is an integral curve of ψ ,

(b') $\|\dot{\psi} - \dot{\phi}\| < \delta'$.

So we have

(i) $\psi_{\sum_{i=0}^{n-1} t_i} x'_0 = x'_n,$

(ii) $d(\psi_t, \phi_t) < \delta$ for all $t \in [0, 2]$.

From definition of topological stability, there exists $h: M \rightarrow M$ continuous such that

$$d(h, I) < \varepsilon/2, h(\text{orbit of } \psi) \subseteq (\text{orbit of } \phi).$$

Take the point hx'_0 , there is a continuous map $\sigma_{x'_0}: [0, t_0] \rightarrow \mathbb{R}$

with $\sigma_{x'_0}(0) = 0$ such that

$$\phi_{\sigma_{x'_0}(t)} h x'_0 = h \psi_t x'_0, \quad 0 \leq t \leq t_0.$$

Also for hx'_1 , there is a continuous map $\sigma_{x'_1}: [0, t_1] \rightarrow R$ with

$\sigma_{x'_1}(0) = 0$ such that

$$\phi_{\sigma_{x'_1}(t-t_0)} hx'_1 = h\psi_{t-t_0} x'_1, \quad t_0 \leq t \leq t_0 + t_1$$

If we carry on in the same manner, so for the point hx'_n , there is a continuous map $\sigma_{x'_n}: [0, t_n] \rightarrow R$ with $\sigma_{x'_n}(0) = 0$ such that

$$\phi_{\sigma_{x'_n}(t-\sum_{i=0}^{n-1} t_i)} hx'_n = h\psi_{t-\sum_{i=0}^{n-1} t_i} x'_n, \quad \sum_{i=0}^{n-1} t_i \leq t \leq \sum_{i=0}^n t_i$$

Since $1 \leq t_n \leq 2$ for $0 \leq n \leq k-1$ and if $\sum_{i=0}^{n-1} t_i \leq t \leq \sum_{i=0}^n t_i$, then

$$0 \leq t - \sum_{i=0}^{n-1} t_i \leq 2 \text{ and}$$

$$(*) \quad d(\phi_{\sigma_{x'_n}(t-\sum_{i=0}^{n-1} t_i)} hx'_n, \phi_{\psi_{t-\sum_{i=0}^{n-1} t_i}} x'_n) = d(h\psi_{t-\sum_{i=0}^{n-1} t_i} x'_n, \phi_{\psi_{t-\sum_{i=0}^{n-1} t_i}} x'_n)$$

$$\leq d(h\psi_{t-\sum_{i=0}^{n-1} t_i} x'_n, \psi_{t-\sum_{i=0}^{n-1} t_i} x'_n) + d(\psi_{t-\sum_{i=0}^{n-1} t_i} x'_n, \phi_{\psi_{t-\sum_{i=0}^{n-1} t_i}} x'_n)$$

$$< \xi/2 + \delta < \xi < \epsilon''.$$

Therefore,

$$(**) \quad d(\phi_{\sigma_{x'_n}(t - \sum_{i=0}^{n-1} t_i)}^{n-1} \phi_{t - \sum_{i=0}^{n-1} t_i}^{n-1} hx'_n, \phi_{t - \sum_{i=0}^{n-1} t_i}^{n-1} \phi_{t - \sum_{i=0}^{n-1} t_i}^{n-1} x'_n) < \epsilon''.$$

Since $d(hx'_n, x'_n) < \xi$, therefore

$$d(\phi_{t - \sum_{i=0}^{n-1} t_i}^{n-1} \phi_{t - \sum_{i=0}^{n-1} t_i}^{n-1} hx'_n, \phi_{t - \sum_{i=0}^{n-1} t_i}^{n-1} \phi_{t - \sum_{i=0}^{n-1} t_i}^{n-1} x'_n) < \epsilon' \text{ for } \sum_{i=0}^{n-1} t_i \leq t \leq \sum_{i=0}^n t_i, 0 \leq n \leq k.$$

Using this, (**) and lemma 4.1, then

$$|\sigma_{x'_n}(t - \sum_{i=0}^{n-1} t_i) - (t - \sum_{i=0}^{n-1} t_i)| < \epsilon' \text{ for } \sum_{i=0}^{n-1} t_i \leq t \leq \sum_{i=0}^n t_i, 0 \leq n \leq k-1.$$

From this it is clear that there exists a homeomorphism α_n on $[0, t_n]$ with $\alpha_n(0) = 0$, $\alpha_n(t_n) = \sigma_{x'_n}(t_n)$ such that

$$|\alpha_n(t - \sum_{i=0}^{n-1} t_i) - \sigma_{x'_n}(t - \sum_{i=0}^{n-1} t_i)| < \epsilon \text{ for } \sum_{i=0}^{n-1} t_i \leq t \leq \sum_{i=0}^n t_i, 0 \leq n \leq k-1.$$

As we know $\psi_{n-1}^{hx'_0} = x'_n$, $0 \leq n \leq k$, and

$$\phi_{\sigma_{x'_n}(t_n)}^{hx'_n} = h\psi_{t_n}^{x'_n} = hx'_{n+1}, 0 \leq n \leq k-1,$$

so $\phi_{\alpha_n(t_n)}^{hx'_n} = h\psi_{t_n}^{x'_n} = hx'_{n+1}, 0 \leq n \leq k-1.$

Now define $\alpha: [0, \sum_{i=0}^{k-1} t_i] \rightarrow R$ in the following way

$$\alpha(t) = \alpha_n(t - \sum_{i=0}^{n-1} t_i) + \sum_{i=0}^{n-1} \alpha_i(t_i) \text{ for } \sum_{i=0}^{n-1} t_i \leq t \leq \sum_{i=0}^n t_i.$$

α is a homeomorphism onto its image (increasing) Pick

$t \in [0, \sum_{i=0}^{n-1} t_i]$ and without loss of generality assume

$\sum_{i=0}^{n-1} t_i \leq t \leq \sum_{i=0}^n t_i$. From the way we choose the points x'_i and

because $0 \leq t - \sum_{i=0}^{n-1} t_i \leq 2$ we will have

$$\begin{aligned} d(\phi_{\alpha(t)} h x'_0, \phi_{t - \sum_{i=0}^{n-1} t_i} \phi_{\sum_{i=0}^{n-1} \alpha_i(t_i)} h x'_0, \phi_{t - \sum_{i=0}^{n-1} t_i} \phi_{\sum_{i=0}^{n-1} \alpha_i(t_i)} h x'_n) &= d(\phi_{\alpha(t)} h x'_0, \phi_{t - \sum_{i=0}^{n-1} t_i} \phi_{\sum_{i=0}^{n-1} \alpha_i(t_i)} h x'_0, \phi_{t - \sum_{i=0}^{n-1} t_i} \phi_{\sum_{i=0}^{n-1} \alpha_i(t_i)} h x'_n) \\ &= d(\phi_{\alpha_n(t - \sum_{i=0}^{n-1} t_i)} \phi_{\sum_{i=0}^{n-1} \alpha_i(t_i)} h x'_0, \phi_{t - \sum_{i=0}^{n-1} t_i} \phi_{\sum_{i=0}^{n-1} \alpha_i(t_i)} h x'_n) \\ &= d(\phi_{\alpha_n(t - \sum_{i=0}^{n-1} t_i)} h x'_n, \phi_{t - \sum_{i=0}^{n-1} t_i} \phi_{\sum_{i=0}^{n-1} \alpha_i(t_i)} h x'_n) \\ &\leq d(\phi_{\alpha_n(t - \sum_{i=0}^{n-1} t_i)} h x'_n, \phi_{\sigma_{x'_n}(t - \sum_{i=0}^{n-1} t_i)} h x'_n) \\ &\quad + d(\phi_{\sigma_{x'_n}(t - \sum_{i=0}^{n-1} t_i)} h x'_n, \phi_{t - \sum_{i=0}^{n-1} t_i} \phi_{\sum_{i=0}^{n-1} \alpha_i(t_i)} h x'_n) + d(\phi_{t - \sum_{i=0}^{n-1} t_i} \phi_{\sum_{i=0}^{n-1} \alpha_i(t_i)} h x'_n, \phi_{t - \sum_{i=0}^{n-1} t_i} \phi_{\sum_{i=0}^{n-1} \alpha_i(t_i)} h x'_n) \\ &< \varepsilon/3 + \varepsilon'' + \lambda/3 < \varepsilon. \end{aligned}$$

This means $(\{x_n\}_0^k, \{t_n\}_0^k)$ with $1 \leq t_n \leq 2$, $0 \leq n \leq k$ is ε -traced by an orbit of ϕ . Using the corollary of proposition 4.2 ϕ has P.O.T.P. and the proof is complete. \square

Some important questions can be answered here.

Corollary 1.

Assume ϕ is a C^1 expansive flow on a compact manifold M without fixed points, ψ is a flow on M obtained from ϕ by changing velocity. Then ϕ is topologically stable if and only if ψ is topologically stable.

Proof: *Follows directly* from theorem 6,4.

Also in the statement of the above corollary if ϕ and ψ are conjugate with preserved orientation then either both are topologically stable or ~~neither~~ of them is.

Corollary 2.

Assume T is an expansive homeomorphism on a compact manifold M and $\phi_f: M_f \times \mathbb{R} \rightarrow M_f$ its suspension flow under a continuous map $f: M \rightarrow \mathbb{R} > 0$. Then

- (a) ϕ_f is topologically stable $\implies T$ is topologically stable.
- (b) $\dim M \geq 2$, T is topologically stable $\implies \phi_f$ is topologically stable.

Proof:

(a) ϕ_f has no fixed points, so if we assume ϕ_f is topologically stable, then theorem 6 implies that ϕ_f has P.O.T.P. Using theorem 3, T has also P.O.T.P. Since T is expansive ([25], theorem 4) will imply that T is topologically stable (see chapter 1, section 1.5).

(b) Assume $\dim M \geq 2$, T is topologically stable, then by ([25], theorem 11), T has P.O.T.P. Using theorem 3, ϕ_f has P.O.T.P. Now if T is expansive so does ϕ_f ([2], theorem 6). By theorem 4 ϕ has P.O.T.P.

REFERENCES

- [1] BRYANT, B.F., and WALTERS, P. "Asymptotic properties of expansive homeomorphisms". Math. Syst. Theory, Vol. 3, No. 1 (1969).
- [2] BOWEN, R, and WALTERS, P. "Expansive one-parameter flows". Journ. Diff. Eqns. 12. (1972), 180-193.
- [3] BOWEN, R. "Periodic points and measures for Axiom A diffeomorphism". Trans. Amer. Math. Soc. 154, (1971) 377-397.
- [4] BOWEN, R. "Periodic points for hyperbolic flows". Amer. J. Math. **94** (1972), 1-30.
- [5] BOWEN, R. "Topological entropy and Axiom A". Proc. Symp. Pure Math., 14 (1970), 23-41.
- [6] BOWEN, R. "Entropy-expansive maps". Trans. Amer. Math. Soc., 164 (1972), 323-331.
- [7] BOWEN, R. "On Axiom A diffeomorphism", Springer Lecture Notes, Vol. 470 (1975).
- [8] BOWEN, R. "w-limit sets for Axiom A diffeomorphism". J. Diff. Eqns., 18 (1975), 333-339.
- [9] BOWEN, R. "Equilibrium states and the ergodic theory of Axiom A diffeomorphism". Springer Lecture Notes, Vol. 470 (1975).

- [10] CHURCHILL, R.C. "A geometric criterion for hyperbolicity of flows". To appear.
- [11] DENKER, M., GRILLENBERGER, C., and SIGMUND, C.G.K. "Ergodic theory on compact spaces". Springer Lecture Notes, Vol. 575 (1976).
- [12] FRANKE, J.E. and SELGRADE, J.F. "Hyperbolicity and chain recurrence". J. Diff. Eqns, 26 (1977), 27-36.
- [13] FRANKE, J.E. and SELGRADE, J.F. "Equivalent Criteria for Axiom A and no cycles". To appear.
- [14] FRANKE, J.E. and SELGRADE, J.F. "Hyperbolicity and cycles". Trans. Amer. Math. Soc., 245 (1978), 251-262.
- [15] HUMPHRIES, P.D. "Change of velocity in dynamical systems". J. London Math. Soc., (2) 7 (1974), 747-757.
- [16] LANG, S. "Introduction to differentiable manifolds". New York Interscience Publishers (1962).
- [17] MUNKRES, J.R. "Elementary differential topology", Princeton University Press (1966).
- [18] NITECKI, Z and SHUB, M. "Filtrations, decompositions and explosions", Amer. J. Math, 97 (1976), 1029-1047.
- [19] PARRY, W. "Cocycles and velocity changes", J. London Math. Soc. (2) 5(1972), 511-516.
- [20] RUELLE, D. "Statistical mechanics on a compact set with Z^Y action satisfying expansiveness and specification". Bull. Amer. Math. Soc., Vol. 78, No.6, (1972), 988-991.

- [21] SIGMUND, K. "Generic properties of invariant measures for Axiom A diffeomorphism". Invent. Math., 11 (1970), 99-109.
- [22] SIGMUND, K. "Ergodic averages for Axiom A diffeomorphism". z. Wahrscheinlickeits.theorie verw.Gebiete, 20 (1971), 319.324.
- [23] SIGMUND, K. "Mixing measures for Axiom A diffeomorphisms". Proc. Amer. Math. Soc., 36 (1972), 497-504.
- [24] SIGMUND, K. "On dynamical systems with specification property". Trans. Amer. Math. Soc., 190 (1974), 285-299.
- [25] WALTERS, P. "On the pseudo orbit tracing property and its relationship to stability". Springer Lecture Notes, Vol. 668.
- [26] WALTERS, P. "Introduction Lectures on ergodic theory". Springer Lecture Notes, Vol. 458 (1975).
- [27] WEISS, B. "Topological transitivity and ergodic measures". Math. Syst. Theory, 5 (1971), 71-75.

CHAPTER V

5.1. Introduction

In this chapter we consider the connections between several concepts in dynamics for the important class of transformations called codimension one Anosov diffeomorphism [7]. Codimension 1 toral automorphisms are examples of these. The assumption of codimension one is important to get our results. We say two toral automorphisms A, B are rationally dependent if there are two positive integers k, ℓ such that $A^k = B^\ell$. Work is carried in [4] - [10] and [13] for finding relations between the rational dependence of A, B and the set of normal points with respect to A and B . In this work we study the relations between commutativity of two hyperbolic toral automorphisms A, B and the normal points with respect to A, B and the stable (unstable) manifolds of A, B at the point O of the torus T^n and how these properties are related to each other. In this chapter we start with definitions and notation and some background around this field, and we define an I-hyperbolic toral automorphism whose matrix of integers has irreducible characteristic polynomial over the field of rational numbers. An example of such an automorphism is a codimension 1 hyperbolic toral automorphisms of T^n . Such an automorphism has no multiple eigenvalues (lemma 5.1) and also we show that two I-hyperbolic toral automorphisms of T^n commute if and only if they have a common eigenvector. Moreover, A, B have a common eigenvector if and only if all the eigenvectors are in common if and only if A, B commute. In section 5.5 we start with definition and background about *uniform distribution modulo 1* (*u.d.mod 1*) and we prove that

the sequence (tX_n) , $n = 1, 2, \dots$ of complex numbers with $\|X_m - X_n\| \geq K > 0$ for $m \neq n$ is u.d. mod 1 for almost all complex t . From this we prove that if C is an expanding eigenvector of a hyperbolic automorphism $A: T^n \rightarrow T^n$, then $\pi R(tc) \in N(A)$ ($N(A)$ is the set of normal points and πR natural projection from C^n onto T^n). If W_A^S denotes the stable manifold of $A: T^n \rightarrow T^n$ at O , then in theorem one, we proved that for two I-hyperbolic toral automorphisms A, B of T^n , $N(A) = N(B) \implies W_A^S = W_B^S$. Moreover, in the statement of theorem one if A, B are codimension 1, the $N(A) = N(B) \implies W_A^S = W_B^S$, $W_A^u = W_B^u$ and $AB = BA$. Also in theorem 2 we prove for any two I-hyperbolic toral automorphisms of T^n , the following holds:

- (a) $AB = BA \implies AN(B) = N(B)$,
- (b) $AN(B) = N(B) \implies AW_B^S = W_B^S$.

Also in the statement of theorem 2 if A, B are codimension 1, then the following are equivalent

- (a) A, B commute.
- (b) The set of normal points with respect to B is invariant under A .
- (c) The set of normal points with respect to A is invariant under B .
- (d) The stable (unstable) manifold with respect to B is invariant under A .

Finally some applications are carried out and we sketch the idea of how one can define normal points with respect to codimension 1 Anosov diffeomorphism f on a compact connected C^∞ manifold. Also in this chapter we show by using theorem 3 that for any I-hyperbolic toral automorphism A of T^n we have $N(A) \neq N(A^{-1})$. So for two ergodic automorphisms A, B of T^2 to be rationally dependent we need not only a common eigenvector but also we need that $W_A^S = W_B^S$ or $W_A^u = W_B^u$ see proposition 5.1.

5.2. Notation and Definitions.

Let X be a compact metric space and S a continuous surjective map from X to itself and $\mathcal{C}(X)$ the space of continuous functions on X into the space of complex numbers. A point $x \in X$ is said to be quasi-regular point for the map S if

$$f^*(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} f(S^j(x))$$

exists for all f belonging to $\mathcal{C}(X)$. Let $Q(S)$ denote the set of quasi-regular points for the map S and $M(S)$ the space of normalized S -invariant Borel measures on X , together with the weak topology of measures on X [4]. The point x is said to be a generic point for the measure μ if the map $f \rightarrow f^*(x)$ on $\mathcal{C}(X)$ defines the measure μ in $M(S)$ (see [9]).

If, in particular, we have a group G , an endomorphism $A; G \rightarrow G$ and m , the Haar measure which is invariant under A , then x is said to be a normal point with respect to $A, (x \in N(A))$ if x is a generic point for m (see [3]).

Let $T^n = R^n/Z^n$ denote the n -dimensional torus or $T^n = S^1 \times S^1 \times \dots \times S^1$, (n times) where R^n is n -space of real numbers and Z^n is the subgroup of R^n consisting of points with integer coordinates and S^1 is the 1-dimensional circle. So T^n is the set of n -tuples of reals mod. 1. T^n is a compact abelian group with respect to addition mod. 1, and here the Haar measure is just the Lebesgue measure.

If $[A] = (a_{ij})$ is an element of $GL(m;Z)$, or in other words an $m \times m$ -matrix with integer entries and determinant ± 1 , then $[A]$ is a linear automorphism of R^m sending Z^m onto Z^m . So this means that $[A]$ defines an automorphism A on T^m and conversely any automorphism of T^m induces such an $m \times m$ -matrix.

Now let A be an automorphism of T^n . Then A is ergodic if and only if A has no eigenvalues which are roots of unity (see [4] , [13]). Now, as we know some or probably all of the eigenvalues are complex numbers, so here we can consider the toral automorphism A as a linear automorphism \bar{A} of \mathbb{C}^n in the following way:

For $(x_1, x_2, \dots, x_n) \in \mathbb{C}^n$

$$\bar{A}(x_1, x_2, \dots, x_n) = [A](R(x_1), R(x_2), \dots, R(x_n)) + i[A](I(x_1), I(x_2), \dots, I(x_n))$$

where $[A]$ is the $n \times n$ -matrix of A and R, I are the real and imaginary parts of a complex number. Alternatively, a better impression may be given by this commutative diagram,

$$\begin{array}{ccc}
 \mathbb{C}^n & \xrightarrow{\bar{A}} & \mathbb{C}^n \\
 \downarrow R & & \downarrow R \\
 \mathbb{R}^n & \xrightarrow{[A]} & \mathbb{R}^n \\
 \downarrow \pi & & \downarrow \pi \\
 \mathbb{T}^n & \xrightarrow{A} & \mathbb{T}^n
 \end{array}$$

where R is a function defined on n -tuples by

$$R(x_1, x_2, \dots, x_n) = (R(x_1), R(x_2), \dots, R(x_n))$$

and π is the natural projection from \mathbb{R}^n onto $\mathbb{R}^n/\mathbb{Z}^n$.

An element c in \mathbb{C}^n is called an eigenvector of the toral automorphism A if there is a complex number λ such that $\bar{A}(c) = \lambda c$.

A linear automorphism v of a finite dimensional vector space E over the field of complex number (or real numbers) will be called hyperbolic if its eigenvalues λ_i satisfy $|\lambda_i| \neq 1$ for all i (see [12]), the eigenvector c of v is called expanding if the corresponding eigenvalue is greater than one in absolute value, and contracting if less than one in absolute value.

Hence for a hyperbolic map $v: E \rightarrow E$ we have a canonical invariant (under v) splitting of E , $E = E^S \oplus E^U$ (direct sum) where E^S is the eigenspace generated by the contracting eigenspace of v and E^U the eigenspace generated by the expanding eigenspace of v . Thus v , restricted to E^S is contracting and to E^U expanding. In fact the hyperbolic elements of the general linear group $GL(E)$ are open and dense.

5.3. Anosov Diffeomorphisms and Stable (Unstable) Manifolds.

Let X be a space and E a Riemannian vector bundle over X such that each fibre E_x is equipped with an inner product \langle, \rangle_x in a continuous manner. This allows one to speak of the norm $\|v\|$ of a vector $v \in E_x$. A bundle map between vector space bundles is a fibre preserving map $\phi: E \rightarrow E$ of a Riemannian vector space bundle into itself and also is called contracting if there exists $c > 0$, $0 < \lambda < 1$ such that for all $v \in E$, and $m \in \mathbb{Z}^+$

$$\|\phi^m(v)\| \leq c\lambda^m \|v\|$$

and called expanding if there exists $d > 0$, $\mu > 1$ such that for all $v \in E$, and $m \in \mathbb{Z}^+$

$$\|\phi^m(v)\| \geq d\mu^m \|v\|$$

(see [12]).

The following are known facts. [12].

- (a) If X is compact, then the property of contracting or expanding for $\phi: E \rightarrow E$, (E is a Riemannian vector space bundle over X) is independent of the choice of Riemannian metric.
- (b) The inverse of a contracting bundle automorphism is an expanding bundle automorphism and vice-versa.

Let M be a manifold with a diffeomorphism $f: M \rightarrow M$, also let d be a metric on M and $P \in M$ be a hyperbolic fixed (i.e. $T_P M$ has a continuous splitting $T_P M = E^s + E^u$ invariant under the derivative Df such that Df is contracting on E^s and expanding on E^u .) point of f . Then define

$$W_f^S(P) = \{y; y \in M: d(f^n(P), f^n(y)) \rightarrow 0 \text{ as } n \rightarrow \infty\},$$

$$W_f^u(P) = \{y; y \in M: d(f^n(P), f^n(y)) \rightarrow 0 \text{ as } n \rightarrow -\infty\}.$$

$W_f^S(P)$ and $W_f^u(P)$ are called stable and unstable manifolds at P respectively. If $M = G$, a Lie group, then W_f^S, W_f^u will denote the stable, unstable manifold at the zero element of G . Now for $f: M \rightarrow M$ with $T_p M = V^S + V^u$ (i.e., f is an Anosov diffeomorphism [1]) there exists a contraction $g: W_f^S(P) \rightarrow W_f^S(P)$ with fixed point P_0 and an injective equivariant immersion $J: W_f^S(P) \rightarrow M$ such that $J(P_0) = P$ and $DJ(P_0): T_{P_0}(W_f^S(P)) \rightarrow T_p M$ is an isomorphism onto V^S .

Definition.

A toral automorphism $A: T^n \rightarrow T^n$ will be called I-hyperbolic if

- (a) $\bar{A}: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is hyperbolic (i.e., none of the eigenvalues of \bar{A} have unit absolute value.)
- (b) The matrix of A , $[A]$ has irreducible characteristic polynomial over the field of rational numbers.

It is clear that this automorphism A is an Anosov diffeomorphism and so is ergodic. We say that A is of "codimension one" if its stable or unstable manifold has dimension one. To know more about the stable and unstable manifold we need to prove the following:

Lemma 5.1.

Let $K[x]$ be the polynomial domain over a field K (i.e., coefficients of the polynomial are elements of K [16]). Then an irreducible polynomial $f(x) \in K[x]$ such that $f'(x) \neq 0$ has all its roots of multiplicity one. (where K is a field of rational numbers and $f'(x)$ the derivative of $f(x)$).

Proof: The divisors of $f(x)$ are the units of $K[x]$ together with those elements of $K[x]$ of the form $\alpha f(x)$, α is a unit of $K[x]$. Since the units of $K[x]$ are $k \setminus 0$, it follows that the common divisors of $f(x)$ and $g(x)$ are units for every other polynomial $g(x) \in K[x]$ in which the degree of $g(x)$ is less than the degree of $f(x)$. Since $K[x]$ is a ^{Euclidean} principal ideal domain it follows that there exists $a(x), b(x)$ in $K[x]$ such that

$$a(x)f(x) + b(x)g(x) = 1.$$

This implies that every root of $f(x)$ is not a root of $g(x)$ because if u is a root of $f(x)$ and $g(x)$, then we have

$$0 = a(u)f(u) + b(u)g(u) = 1$$

which is a contradiction. Now to show all roots of $f(x)$ are of multiplicity one, suppose

$$f(x) = (x-u)^m h(x)$$

where u is a root of $f(x)$ and $h(x) \in K[x]$. Therefore

$$f'(x) = m(x-u)^{m-1}h(x) + (x-u)^m h'(x),$$

now if $m > 1$, we have $f'(u) = 0$ and this is again a contradiction because $f'(u)$ should be not zero. Therefore $m = 1$. \square

So from the above lemma any toral I-hyperbolic automorphism of T^n has n -distinct eigenvalues λ_i ; $i = 1, 2, \dots, n$ corresponding to n -distinct eigenvectors C_i , $i = 1, 2, \dots, n$ which generated \mathbb{C}^n and they are the only eigenvectors for such an automorphism. So if A is an I-hyperbolic automorphism of T^n and c_1, c_2, \dots, c_k are the contracting eigenvectors and $c_{k+1}, c_{k+2}, \dots, c_n$ are the expanding eigenvectors, then the stable and unstable manifold at O of A are

$$W_A^S = \pi R\{t_1 c_1 + t_2 c_2 + \dots + t_k c_k; t_i \in \mathbb{C}: 1 \leq i \leq k\}.$$

and

$$W_A^u = \pi R\{t_{k+1} c_{k+1} + t_{k+2} c_{k+2} + \dots + t_n c_n; t_i \in \mathbb{C}: k+1 \leq i \leq n\}$$

i.e., there is a continuous splitting of $\mathbb{C}^n = E^S \oplus E^u$ such that each of E^S and E^u are invariant under \bar{A} and $W_A^S = \pi R(E^S)$ and $W_A^u = \pi R(E^u)$. So when we say A is of codimension one this means \bar{A} has only one contracting eigenvector or only one expanding eigenvector.

Remark 5.1.

It is a fact that if A is a hyperbolic toral automorphism of T^n and A is of codimension one, then A is an I-hyperbolic automorphism. For, suppose it is not true (i.e., the matrix of A ,

[A] has a reducible characteristic polynomial) say the characteristic polynomial of [A] is equal to

$$f(x).g(x) = (x^s + a_1x^{s-1} + \dots + a_s)(x^t + b_1x^{t-1} + \dots + b_t)$$

where a_s and a_t are integers and $s + t = n$. Since A is of codimension one and if we assume the dimension of W_A^u to be one, then either $|a_s| < 1$ or $|b_t| < 1$. Contradicting the fact that a_s and a_t are non zero integers. In the case when dimension of W_A^u is $(n-1)$, use the same idea as above for A^{-1} . Therefore [A] has irreducible characteristic polynomial, this means A is an I-hyperbolic automorphism.

Lemma 5.2.

Let A and B be two I-hyperbolic toral automorphisms of T^n . Then A and B commute ($AB = BA$) if and only if \bar{A} and \bar{B} have the same eigenvectors.

Proof: Assume $AB = BA$. In order to show that \bar{A} and \bar{B} also commute, let $h = \bar{A}\bar{B} - \bar{B}\bar{A}$ be a continuous function on \mathbb{T}^n such that $h(x) \in \mathbb{Z}^n \times \mathbb{Z}^n$ for every point $x \in \mathbb{T}^n$ where \mathbb{Z}^n is an n-product of the set of integers. Since \mathbb{T}^n is connected and $h(0) = 0$, the image of \mathbb{T}^n under h should be zero (i.e., h is a zero map). This means $\bar{A}\bar{B} = \bar{B}\bar{A}$. If c is an eigenvector of \bar{A} with eigenvalue λ , then $\bar{A}\bar{B}(c) = \bar{B}\bar{A}(c) = \lambda\bar{B}(c)$. This means that $\bar{B}(c)$ is also an eigenvector of \bar{A} with same eigenvalue λ . Lemma 5.1 implies that $\bar{B}(c) = tc$, where $t \in \mathbb{C}$.

Conversely. Assume \bar{A} and \bar{B} have the same eigenvectors c_1, c_2, \dots, c_n with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and $\mu_1, \mu_2, \dots, \mu_n$ with respect to \bar{A} and \bar{B} respectively. Then $\bar{A}\bar{B}(c_i) = \mu_i \lambda_i c_i = \bar{B}\bar{A}(c_i)$ for $1 \leq i \leq n$ and since these eigenvectors generated \mathbb{C}^n , $\bar{A}\bar{B} = \bar{B}\bar{A}$ on \mathbb{C}^n and this means $AB = BA$.

Lemma 5.3.

Let A be an I-hyperbolic toral automorphism of T^n . Then the subgroup G generated by any eigenvector of A ($G = \pi R\{tc; t \in \mathbb{C}\}$, c is an eigenvector of \bar{A}) is dense in T^n .

Proof: Let \bar{G} be the closure of G . Since \bar{G} is closed and connected subgroup of T^n . \bar{G} is also a torus say $\bar{G} = T^k$, G is also invariant under A , therefore the restriction map $A/\bar{G}: T^k \rightarrow T^k$ has c as an eigenvector of A/\bar{G} with eigenvalue λ . So λ is a root of a polynomial of degree k but $[A]$ has an irreducible characteristic polynomial of degree n with λ as a root. As a result $k = n$, implies $\bar{G} = T^n$. \square

From these two above lemmas 5.2, 5.3, one can say that for any two I-hyperbolic toral automorphisms A, B of T^n either all the eigenvectors of \bar{A} and \bar{B} are in common or no common eigenvector occurs. Also A and B commute if and only if there is one common eigenvector.

5.4. Measures and Weak Topology

Given a metric space X , a homeomorphism $s:X \rightarrow X$ and a positive integer N , consider x a point in X and let $\mu_{N,x}$ denote the measure given by

$$\int f d\mu_{N,x} = \frac{1}{N} \sum_{j=0}^{N-1} f(s^j x),$$

for $f \in \mathcal{C}(X)$. Let $V^S(x)$ denote the set of accumulation points of $\mu_{N,x}$ in the weak topology for measures. It is easy to see that $V^S(x)$ is a non-empty closed and connected subset of $M(S)$. ($M(S)$ space of all measures). It can be said that $V^S(x)$ reduces to a point if and only if x is a quasi-regular point (see [11]).

Lemma 5.4. Let $y \in X$ satisfy $\lim_{j \rightarrow \infty} d(s^j x, s^j y) = 0$ for some metric d on X . Then $V^S(x) = V^S(y)$.

Proof: (see [11]).

Lemma 5.5.

Let A be an I -hyperbolic toral automorphism of T^n and assume y is a point in \mathbb{C}^n such that $\pi R(y) \in W_A^S$, then for every point $x \in \mathbb{C}^n$ if $w = ty + x$, $t \in \mathbb{C}$, then $V^A(\pi R(w)) = V^A(\pi R(x))$.

Proof: Let c_1, c_2, \dots, c_k be the contracting eigenvectors of A with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$. Since $\pi R(y) \in W_A^S$, we have $y = r_1 c_1 + r_2 c_2 + \dots + r_k c_k$ where $r_i \in \mathbb{C}$ and $1 \leq i \leq k$.

Hence

$$d(A^n \pi R(w), A^n \pi R(x)) \leq b_1 t |\lambda_1|^n + b_2 t |\lambda_2|^n + \dots + b_k t |\lambda_k|^n$$

where b_1, b_2, \dots, b_k are elements depending on the choice of c_1, c_2, \dots, c_k and the metric d on \mathbb{T}^n . $|\lambda_i| < 1$ for all $i = 1, 2, \dots, k$ implies that $\lim_{n \rightarrow \infty} d(A^n \pi R(w), A^n \pi R(x)) = 0$. Using lemma 5.4 we have

$$V^A(\pi R(w)) = V^A(\pi R(x)). \quad \square$$

From the above lemma it follows that if x is a normal point with respect to an I-hyperbolic toral automorphism $A: T^n \rightarrow T^n$ and $y \in W_A^S$, then $\pi R(t\bar{y} + \bar{x})$ is a normal point of A where $\bar{x}, \bar{y} \in \mathbb{T}^n$ of the fibres $(\pi R)^{-1}(x), (\pi R)^{-1}(y)$ respectively. From now on we use \bar{x} as a standard notation for a point in the fibre of x .

5.5. Uniform Distribution Modulo 1.

For a real number x , let $[x]$ denote the integral part of x . (i.e., the greatest integer $\leq x$). Let $\{x\} = x - [x]$ be the fractional part of x , or the residue of x modulo 1.

Let $w = (x_n)$, $n = 1, 2, 3, \dots$ be a given sequence of real numbers. For a positive integer N and a subset E of $I = [0, 1]$, Let $A(E; N; w)$ be defined as a number of terms x_n , $1 \leq n \leq N$, for which $\{x_n\} \in E$.

Definition.

The sequence $w = (x_n)$, $n = 1, 2, \dots$ of real numbers is said to be uniformly distributed modulo 1 (u.d. mod.1) if for every pair a, b of real numbers with $0 \leq a < b \leq 1$ we have

$$\lim_{N \rightarrow \infty} \frac{(A[a, b]; N; w)}{N} = b - a.$$

The notion of uniform distribution is due to H. Weyl [15]. Also see [7]. Koksma [6] has shown an important theorem called "Koksma's General Metric Theorem".

Theorem.

Let $(U_n(x))$, $n = 1, 2, \dots$ be a sequence of real numbers defined for every x in an interval $[a, b]$ and for every $n \geq 1$, let $U_n(x)$ be continuously differentiable on $[a, b]$ and suppose that for any two positive integers m, n ($m \neq n$), the function $U'_m(x) - U'_n(x)$ is monotone with respect to x and that $|U'_m(x) - U'_n(x)| \geq K > 0$, where K does not depend on x, m , and n . Then the sequence $(U_n(x))$, $n = 1, 2, \dots$ is u.d. mod 1 for almost all x in $[a, b]$.

Proof: Koksma [6].

Corollary

If (λ_n) , $n = 1, 2, \dots$ is a sequence of real numbers with $|\lambda_m - \lambda_n| \geq \delta > 0$ for $m \neq n$ and δ a positive constant, then the

sequence $(\lambda_n x)$ is u.d. mod 1 for almost all real numbers x .

Uniform distribution modulo 1 in R^S (i.e., the multi-dimensional case) was first considered by Weyl [15] , [7].

Let $a = (a_1, a_2, \dots, a_s)$ and $b = (b_1, b_2, \dots, b_s)$ be two vectors of R^S . If $a_j < b_j$ ($a_j \leq b_j$) for $j=1, 2, \dots, s$, the set of points $X \in R^S$ such that $a_j \leq x_j < b_j$ will be denoted by $[a, b)$. The s -dimensional unit cube I^S is the interval $[0, 1)$, where $0 = (0, 0, \dots, 0)$ and $1 = (1, 1, \dots, 1)$. The integral part of $X = (x_1, x_2, \dots, x_s)$ is $[X] = ([x_1], [x_2], \dots, [x_s])$ and the fractional part of X is $\{X\} = (\{x_1\}, \{x_2\}, \dots, \{x_s\})$. Let (X_n) , $n = 1, 2, \dots$, be a sequence of vectors in R^S . For a subset E of I^S , $A(E; N)$ denote the number of points $\{X_n\}$, $1 \leq n \leq N$ that lie in E . Then the sequence (X_n) , $n = 1, 2, \dots$, is said to be u.d. mod 1 in R^S if

$$\lim_{N \rightarrow \infty} \frac{A([a, b); N)}{N} = \prod_{j=1}^s (b_j - a_j)$$

for all intervals $[a, b) \subseteq I^S$.

In the case when $(X_n = \alpha_n + i\beta_n)$, $n = 1, 2, \dots$ is a sequence of complex numbers, we say (X_n) $n = 1, 2, \dots$ is u.d. mod 1 if (α_n, β_n) , $n = 1, 2, \dots$ is u.d. mod 1 in R^2 .

Lemma 5.6.

Let K be a positive constant and let (X_n) , $n = 1, 2, \dots$, be a sequence of complex numbers with $\|X_m - X_n\| \geq K$ for $m \neq n$. Then the sequence (tX_n) , $n = 1, 2, \dots$, is u.d. mod 1 for almost all complex number t .

Proof: The proof of this lemma depends on the following two results:

(a) (Cf.[7] theorem 6.3) A sequence (X_n) , $n = 1, 2, \dots$, is u.d. mod 1 in R^S if and only if for every lattice point $h \in \mathbb{Z}^n$, $h \neq 0$, the sequence of real numbers $(\langle h, X_n \rangle)$, $n = 1, 2, \dots$, is u.d. mod 1, where \langle, \rangle is the inner product.

(b) (see [2]) If $(\alpha_n + \beta_n)$, $n = 1, 2, \dots$, is a sequence of real numbers with $(\alpha_m - \alpha_n)^2 + (\beta_m - \beta_n)^2 \geq K > 0$ where K is a positive constant, then $r\alpha_n + s\beta_n$ is u.d. mod 1 for almost all real numbers r and s .

Now to prove the lemma, let $X_n = \alpha_n + i\beta_n$ and $t = r + is$.

$$tX_n = r\alpha_n - s\beta_n + i(r\beta_n + s\alpha_n).$$

Let $h = (M, N) \in \mathbb{Z}^2$, $h \neq 0$ so we have

$$M(r\alpha_n - s\beta_n) + N(r\beta_n + s\alpha_n) = r(M\alpha_n + N\beta_n) + s(-M\beta_n + N\alpha_n).$$

To show such a sequence is u.d. mod 1 for almost all real numbers r and s take

$$\begin{aligned} & (M\alpha_m + N\beta_m - M\alpha_n - N\beta_n)^2 + (-M\beta_m + N\alpha_m + M\beta_n - N\alpha_n)^2 \\ &= M^2(\alpha_m - \alpha_n)^2 + M^2(\beta_m - \beta_n)^2 + N^2(\beta_m - \beta_n)^2 + N^2(\alpha_m - \alpha_n)^2 \\ &= (M^2 + N^2)(\alpha_m - \alpha_n)^2 + (M^2 + N^2)(\beta_m - \beta_n)^2 \\ &= (M^2 + N^2)((\alpha_m - \alpha_n)^2 + (\beta_m - \beta_n)^2) \geq K > 0. \end{aligned}$$

Hence (tX_n) , $n = 1, 2, \dots$ is u.d. mod 1 for almost all complex number t . \square

Corollary.

If λ is a complex number and $||\lambda|| > 1$, then $(t\lambda^n)$, $n = 1, 2, \dots$, is u.d. mod 1 for almost all complex numbers t .

If λ is a real number and $|\lambda| > 1$, then it is clear that $(t\lambda^n)$, $n = 1, 2, \dots$, is also u.d. mod 1 for almost all real numbers t . (Using corollary of Koksma's General Metric Theorem).

5.6. Normal Points

From lemma 5.5 we know that if c is a contracting eigenvector of a hyperbolic toral automorphism A of T^n , then $\pi R(tc)$ cannot be a normal point with respect to A for every $t \in \mathbb{C}$, in fact $V^A(\pi R(tc)) = V^A(0)$. (i.e., $\pi R(tc)$ is a generic point for the atomic measure μ_0 , $\mu_0(\gamma) = 1$ for every character γ on T^n .)

Lemma 5.7.

Let A be any hyperbolic toral automorphism of T^n and assume c is an expanding eigenvector of A . Then $\pi R(tc) \in N(A)$ for almost every complex number t . Where $N(A)$ is the set of normal points with respect to A . (i.e., generic points to the Haar measure.)

Proof: The proof of this lemma depends on two facts given by the Weyl Criterion [15].

(a) A sequence (X_n) , $n = 0, 1, 2, \dots$, of real numbers is u.d. mod 1 in \mathbb{R} if and only if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i k x_n} = 0$$

for every non zero integer k .

(b) If (X_n) , $n = 1, 2, \dots$, is u.d. mod.1 in \mathbb{R}^S , then all s -coordinate sequences are u.d. mod.1 in \mathbb{R} .

Now to prove the lemma, let λ be the corresponding eigenvalue of c . Since $\|\lambda\| > 1$. Using the corollary of lemma 5.6, we know $(t\lambda^j)$, $j = 0, 1, 2, \dots$, is u.d. mod 1 for almost all complex numbers t . Using a, b above,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} e^{2\pi i R(t\lambda^j)} = 0$$

where $R(t\lambda^j)$ is the real part of $t\lambda^j$. Now let Λ be a set of all non-zero lattice points $h \in \mathbb{Z}^n$ of integers and for every such lattice point $h \neq 0$ define A_h to be a set of complex numbers t such that each of which satisfies the property

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} e^{2\pi i R(t\lambda^j \langle h, c \rangle)} = 0$$

where $\langle h, c \rangle$ means the inner product and $R(t\lambda^j \langle h, c \rangle)$ the real part of $t\lambda^j \langle h, c \rangle$. Λ is a countable set, so the union of all complements of A_h , $h \in \Lambda$ has zero measure. Let A be the intersection of all A_h , $h \in \Lambda$. Then A has zero measure and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} e^{2\pi i R(t\lambda^j \langle h, c \rangle)} = 0$$

for every $t \in A$ and for every $h \in \Lambda$, $h \neq 0$. Let $\gamma \neq 1$ be any character of T^n (see [13]). It is easy to check that the following diagram

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\bar{\gamma}} & \mathbb{C} \\ \pi R \downarrow & & \downarrow \pi R \\ T^n & \xrightarrow{\gamma} & T \end{array}$$

commutes, where $\bar{\gamma}(x) = \langle h, x \rangle$. $h \in \Lambda$, $h \neq 0$ and h depends on the choice of γ . Therefore we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} \gamma A^j(\pi R(tc)) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} \gamma \pi R(\bar{A}^j(tc)) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} \gamma \pi R(t\lambda^j c) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} \pi R(t\lambda^j \bar{\gamma}(c)) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} e^{2\pi i R(t\lambda^j \langle h, c \rangle)} = 0 \end{aligned}$$

for all $t \in A$, since T^n is a compact abelian group with a Haar measure m and $m(A) = 1$. It follows that $\pi R(tc)$ is a normal point with respect to A for almost all complex numbers t . \square

From the above lemma is clear that if c is an expanding eigenvector of an I-hyperbolic toral automorphism, then the subgroup $G = \pi R\{tc; t \in \mathbb{C}\}$ of T^n has almost all its elements as normal points with respect to A .

Corollary.

If A and B are I -hyperbolic toral automorphisms of T^n with $AB = BA$ and $W_A^S \neq W_B^S$, then $N(A) \neq N(B)$, where W_A^S, W_B^S are the stable manifolds of A and B at zero.

Proof: Since $AB = BA$, all the eigenvectors of \bar{A} are also eigenvectors of B and vice versa. But $W_A^S \neq W_B^S$, there is a contracting eigenvector c with respect to A which is expanding with respect to B . Using the above lemma there is a complex number t such that $\pi R(tc) \in N(B)$, but c is contracting with respect to A . This means for every complex number t we have $V^A(\pi R(tc)) = V^A(0)$, implying $N(A) \neq N(B)$. \square

Lemma 5.8.

Let A, B be codimension 1 hyperbolic toral automorphisms of T^n and assume $AB \neq BA$. Then $W_A^S \neq W_B^S$.

Proof: Assume $W_A^S = W_B^S$ and assume both W_A^S, W_B^S have dimension one ($\dim W_A^S = \dim W_B^S = 1$), we therefore have a common eigenvector c of A and B . Lemma 5.2 implies that $AB = BA$. Now, for the case $\dim W_A^S = \dim W_B^S = n-1$, the plane $P \subseteq \mathbb{C}^n$ of dimension $(n-1)$ such that $\pi R(P) = W_A^S$ is invariant under \bar{A} and \bar{B} . Hence one can choose an orthogonal vector $v \in \mathbb{C}^n$ to P . (i.e., $\langle v, x \rangle = 0$ for every $x \in P$, where \langle, \rangle is the inner product) so we have

$\langle v, \bar{A}(x) \rangle = 0$ for all $x \in P$ and also $\langle v, \bar{B}(x) \rangle = 0$ for all $x \in P$.

Now let \bar{A}_t and \bar{B}_t be the transpose of \bar{A} and \bar{B} respectively, therefore

$$\langle \bar{A}_t(v), x \rangle = \langle \bar{B}_t(v), x \rangle = 0$$

for every $x \in P$. Since $\dim P = n-1$, we have v is an eigenvector of \bar{A}_t and \bar{B}_t . Therefore $A_t B_t = B_t A_t$ implies that $(BA)_t = (AB)_t$ and so we have $AB = BA$ and this contradicts the hypothesis that $AB \neq BA$. \square

Theorem 1.

Let A, B be two I-hyperbolic toral automorphisms of T^n . Then $N(A) = N(B)$ implies $W_A^S = W_B^S$.

Proof: Assume $W_A^S \neq W_B^S$, we have two possible cases. Firstly when A, B commute, by the corollary of lemma 5.7. we have $N(A) \neq N(B)$ which contradicts the hypothesis. Secondly, let w be a vector in \mathbb{C}^n such that $\pi R(w) \in W_B^S$ and $\pi R(w) \notin W_A^S$ and let c_1, c_2, \dots, c_k be the contracting eigenvectors with respect to A and P be the subspace of \mathbb{C}^n generated by the set $\{c_1, c_2, \dots, c_k, w, A(w), A^2(w), \dots\}$. Then P is invariant under A and of dimension greater than k . Thus there exists an expanding eigenvector of A such that $c \in P$ and $c \neq c_j, j = 1, 2, \dots, k$. c can be written as a linear combination in the following way

$$c = r_1 c_1 + r_2 c_2 + \dots + r_k c_k + t_1 w + t_2 A(w) + \dots + t_m A^{m-1}(w)$$

where $r_i, t_j \in \mathbb{C}$, $1 \leq i \leq k$, $1 \leq j \leq m$. Using lemma 5.7, there exists a complex number such that $\pi R(tc) \in N(A)$. Now if we assume $N(A) = N(B)$ implies $\pi R(tt_1 w + tt_2 A(w) + \dots + tt_m A^{m-1}(w)) \in N(A)$ because c_1, c_2, \dots, c_k are the contracting eigenvectors of A . But $N(A) = N(B)$ and w is a contracting eigenvector of B , so

$$\pi R(tt_2 A(w) + \dots + tt_m A^{m-1}(w)) \in N(B) = N(A).$$

From now on and without loss of generality we will write the above as follows:

$$tt_2 A(w) + tt_3 A^2(w) + \dots + tt_m A^{m-1}(w) \in N(B) = N(A),$$

$$\text{so } A^{-1}(tt_2 A(w) + tt_3 A^2(w) + \dots + tt_m A^{m-1}(w)) \in N(A) = N(B)$$

w is a contracting eigenvector of B , so

$$tt_3 A(w) + \dots + tt_m A^{m-2}(w) \in N(A) = N(B).$$

If we continue in the same manner, we will have $tt_m w \in N(B)$ and this is a contradiction because $w \in W_B^S$. \square

Corollary.

If A and B are codimension 1 hyperbolic toral automorphisms of T^n , then $N(A) = N(B)$ implies that $W_A^S = W_B^S$, $W_A^u = W_B^u$, and $AB = BA$.

Proof: Obvious from the above theorem and lemma 5.8.

Theorem 2.

For any two I-hyperbolic toral automorphisms A, B of T^n the following holds,

- (a) $AB = BA$ implies $AN(B) = N(B)$ (normal points of B is invariant under A).
- (b) $AN(B) = N(B)$ implies $AW_B^S = W_B^S$ (the stable manifold of B at zero is invariant under A).

Proof: Let $x \in N(B)$ and $\gamma \neq 1$ be any character. Then

$$\frac{1}{N} \sum_{j=0}^{N-1} \gamma B^j A(x) = \frac{1}{N} \sum_{j=0}^{N-1} \gamma AB^j(x) \rightarrow 0 \text{ as } N \rightarrow \infty,$$

therefore $A(x) \in N(B)$. To prove (b) assume $AW_B^S \neq W_B^S$ and assume c_1, c_2, \dots, c_k are the contracting eigenvectors of B . It is clear that there exists j , $1 \leq j \leq k$ such that $A(c_j) \notin W_B^S$. Let H be the subgroup of T^n generated by the set $\{c_1, c_2, \dots, c_k, A(c_j), BA(c_j), B^2A(c_j), \dots\}$, hence $\dim H > K$ and H is invariant under B . So there exists an expanding eigenvector c of B such that

$$c = t_1 c_1 + t_2 c_2 + \dots + t_k c_k + r_1 A(c_j) + r_2 BA(c_j) + \dots + r_m B^{m-2} A(c_j).$$

Using lemma 5.7, there is a complex number t such that $tc \in N(B)$, but c_1, c_2, \dots, c_k are the contracting eigenvectors, therefore

$$\text{tr}_1 A(c_j) + \text{tr}_2 BA(c_j) + \dots + \text{tr}_m B^{m-2} A(c_j) \in N(B).$$

But $A^{-1}N(B) = N(B)$, consequently

$$\text{tr}_1 c_j + \text{tr}_2 A^{-1}BA(c_j) + \dots + \text{tr}_m A^{-1}B^{m-2}A(c_j) \in N(B)$$

$c_j \in W_B^S$ implies again

$$\text{tr}_2 A^{-1}BA(c_j) + \dots + \text{tr}_m A^{-1}B^{m-2}A(c_j) \in N(B)$$

Again $AN(B) = N(B)$, so

$$\text{tr}_2 BA(c_j) + \dots + \text{tr}_m B^{m-2}A(c_j) \in N(B)$$

and hence

$$\text{tr}_2 c_j + \dots + \text{tr}_m A^{-1}B^{m-3}(c_j) \in N(B).$$

$c_j \in W_B^S$ implies $\text{tr}_3 A^{-1}(c_j) + \dots + \text{tr}_m A^{-1}B^{m-3}A(c_j) \in N(B)$.

Now if we repeat this in the same manner as above we will have $\text{tr}_m c_j \in N(B)$ and this is a contradiction. Hence

$$AW_B^S = W_B^S. \quad \square$$

Corollary.

If A, B are two codimension 1 toral hyperbolic automorphisms of T^n , then the following are equivalent:

- (a) A and B commute.
- (b) The set of normal points with respect to B is invariant under A .

- (c) The set of normal points with respect to A is invariant under B.
- (d) The stable manifold at zero with respect to B is invariant under A (i.e., $AW_B^S = W_B^S$)
- (e) The unstable manifold at zero with respect to B is invariant under A. (i.e., $AW_B^u = W_B^u$).

Proof: If $AB = BA$ using the theorem above we have $AN(B) = N(B)$. Conversely, assume $AN(B) = N(B)$. From (b) of the above theorem implies $AW_B^S = W_B^S$. In the case of codimension 1 if $\dim W_B^S = 1$ and c is the only eigenvector of W_B^S implies that c is an eigenvector for A. So it follows that $AB = BA$. Now for the other case (i.e., $\dim W_B^S = n-1$), we have W_B^S is $(n-1)$ -plane in \mathbb{C}^n which is invariant under A and B. Using a similar idea as lemma 5.8 we will have $AB = BA$, this proves $a \iff b$, the rest follows easily.

5.7. Applications

Let $f:M \rightarrow M$ be a codimension 1 Anosov diffeomorphism on a compact connected C^∞ manifold without boundary.

Theorem (Cf. [5] Franks)

Every codimension 1 Anosov diffeomorphism $f:M \rightarrow M$ with $NW(f) = M$ is topologically conjugate to a hyperbolic toral automorphism, $NW(f)$ denotes the set of non-wandering points of f

which defined by $NW(f) = \{x \in M: \text{for any neighbourhood } U \text{ of } x \text{ there is a positive integer } n \text{ such that } f^n U \cap U \neq \emptyset\}$.

Theorem. (Cf [8], Newhouse, theorem 1.2)

Let $f:M \rightarrow M$ be any codimension 1 Anosov diffeomorphism.
Then $NW(f) = M$.

From these two above theorems every Anosov diffeomorphism of codimension 1 is conjugate to a hyperbolic toral automorphism A . (i.e., there is a homeomorphism $\phi:M \rightarrow T^n$ such that the following diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \downarrow \phi & & \downarrow \phi \\ T^n & \xrightarrow{A} & T^n \end{array}$$

is commutative.)

Now we can show that the notion of normal point with respect to a codimension 1 Anosov diffeomorphism of M can be given as follows.

Definition.

A point $x \in M$ is said to be normal point with respect to f if $\phi(x)$ is a normal point with respect to A . (see the above diagram.)

In order to show this definition is independent of the choice of ϕ , T^n we need the following:

Theorem. (Cf. [14] P. Walters, corollary 2 of theorem 2)

Let A, B be two automorphisms of T^n . There exists a non-affine homeomorphism g of T^n such that $gA = Bg$ if and only if the automorphisms A, B are not ergodic.

Assume we have another automorphism $B: T^m \rightarrow T^m$ which also makes the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \downarrow \psi & & \downarrow \psi \\ T^m & \xrightarrow{B} & T^m \end{array}$$

commute. Now assume $\phi(x) \in N(A)$. Using the above theorem we have $\psi\phi^{-1}$ is an affine transformation (i.e., $\psi\phi^{-1} = b.C$) because A and B are ergodic automorphism. Since $B\psi\phi^{-1} = \psi\phi^{-1}A$, therefore $B^j\psi = \psi\phi^{-1}A^j\phi$. Now for a character γ , take

$$\begin{aligned} \frac{1}{N} \sum_{j=0}^{N-1} \gamma B^j \psi(x) &= \frac{1}{N} \sum_{j=0}^{N-1} \gamma (\psi\phi^{-1}) A^j \phi(x) \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \gamma (b.C) A^j \phi(x) = \frac{\gamma(b)}{N} \sum_{j=0}^{N-1} \gamma C A^j \phi(x) \rightarrow 0 \text{ as } N \rightarrow \infty. \end{aligned}$$

This means $\psi(x) \in N(B)$. So the above definition is well defined. (i.e., independent - of the choice of ϕ, T^n).

This definition induces a measure on M with maximal entropy.

In order to show more application we need the following:

Lemma. (Cf. [10]) Let A be an I-hyperbolic toral automorphism of T^n and V a non-empty closed connected subset of $M(A)$. Then there exists $x \in T^n$ such that $V^A(x) = V$.

Proof: Any I-hyperbolic automorphism is an Anosov diffeomorphism and has therefore the specification property. So the lemma follows from theorem 4 in [10].

Theorem 3.

If A is any I-hyperbolic toral automorphism of T^n and V_1, V_2 are two non-empty closed and connected subsets of $M(A)$, then there exists a point $x \in T^n$ such that $V^A(x) = V_1$ and $V^{A^{-1}}(x) = V_2$.

Proof: Let c_1, c_2, \dots, c_k be the contracting eigenvectors of A and $c_{k+1}, c_{k+2}, \dots, c_n$ be the expanding eigenvectors of A . Using the above lemma there exists a point $a \in T^n$ and a point $b \in T^n$ such that $V^A(a) = V_1$ and $V^{A^{-1}}(b) = V_2$. Since c_1, c_2, \dots, c_n are linearly independent vectors, so the following two planes

$$P_1 \quad t_1 c_1 + t_2 c_2 + \dots + t_k c_k + \bar{a}$$

$$P_2 \quad r_{k+1} c_{k+1} + r_{k+2} c_{k+2} + \dots + r_n c_n + \bar{b}$$

(where $\pi R(\bar{a}) = a$ and $\pi R(\bar{b}) = b$) should intersect at least in one point say \bar{x} . There exist s_1, s_2, \dots, s_n are complex numbers such that $\bar{x} = s_1 c_1 + s_2 c_2 + \dots + s_k c_k + \bar{a}$ and $\bar{x} = s_{k+1} c_{k+1} + s_{k+2} c_{k+2} + \dots + s_n c_n + \bar{b}$. Since c_1, c_2, \dots, c_k are the contracting eigenvectors of A and let $x = \pi R(\bar{x})$. Lemma 5.5 implies $V^A(x) = V^A(\pi R(a)) = V_1$.

Since $c_{k+1}, c_{k+2}, \dots, c_n$ are contracting eigenvectors of A^{-1} , then similarly $V A^{-1}(x) = V A^{-1}(\pi R(a)) = V_2$. \square

Corollary.

If A is I -hyperbolic toral automorphism of T^n , then there are uncountably many $x \in T^2$ which are normal with respect to A , but not with respect to A^{-1} .

Proof: One easily sees that $M(A^{-1})$ is an infinite dimensional convex body. There exists uncountably many closed connected subset of $M(A^{-1})$ with more than one element. And the rest follows from the above theorem. \square

Definition.

Two endomorphisms A, B of a group G are said to be rationally dependent if there exist positive (negative) integers k, ℓ such that $A^k = B^\ell$. (see [3])

Theorem. (Cf. [3], Cigler)

If two ergodic endomorphisms A, B are rationally dependent, then every $x \in G$ which is normal with respect to A is also normal with respect to B . (i.e., $N(A) = N(B)$).

In [11] lemma 4. Sigmund stated that "if two ergodic automorphisms A, B of T^2 have a common eigenvector, they are rationally dependent".

Now as we have shown in the corollary of theorem 3 we can have $N(A) \neq N(B)$ while A, B have a common eigenvector (ex. $A = B^{-1}$), so lemma 4 in [11] should be stated in this way. If two ergodic automorphism A, B of T^2 have a common contracting eigenvector (common expanding eigenvector), they are rationally dependent.

Or one can state the following:

Proposition 5.1.

If two ergodic toral automorphism A, B of T^2 commute and $W_A^S = W_B^S (W_A^u = W_B^u)$, then they are rationally dependent.

Proof: Is similar to lemma 4 in [11]. Using the additional condition that $W_A^S = W_B^S (W_A^u = W_B^u)$, one can easily show that the two integers $K, \ell \neq 0$ in which $A^{2K} = B^{2\ell}$ are either both positive or both negative. \square

5.8. References.

- [1] ANOSOV, D.V. "Ergodic properties of geodesic flows on closed Riemannian manifolds of negative curvature", Soviet Math. Dokl. 4(1963), 1153-1156.
- [2] CASSELS, J.W.S. "Some metrical theorems in diophantine approximation". III. Proc. Camb. Phil. Soc. 461 (1950), 219-225.

- [3] CIGLER, J. "Ein gruppentheoretisches"; Analogon zum Begriff der normalen Zahl, J. Reine Angew. Math. 206 (1961), 5-8.
- [4] DENKER, M., GRILLENBERGER, C., and SIGMUND, C.G.K., "Ergodic theory on compact spaces". Springer Lecture Notes, Vol. 575 (1976).
- [5] FRANKS, J. "Thesis University of California", Berkeley (1968).
- [6] KOKSMA, J.F. "Ein mengentheoretischer satz über die Gleichverteilung modulo Ein". Compositio Math. 2 (1935), 250-258.
- [7] KUIPERS, L. and NIEDERREITER, H. "Uniform Distribution of sequences". Wiley, (1974).
- [8] NEWHOUSE, S.E. "On codim.1 Anosov Diffeomorphism". Amer. J. Math., 92(1970), 761-770.
- [9] OXToby, J. "Ergodic sets". Bull. Amer. Math. Soc., 58 (1952), 116-136.
- [10] SIGMUND, K. "Dynamical systems with the specification property". Trans. Amer. Math. Soc., 190 (1974), 285-299.
- [11] SIGMUND, K. "Normal and Quasiregular points for Automorphisms of the torus". Math. System Theory., Vol. 8, No. 3 (1975), 251-255.
- [12] SMALE, S., "Differentiable Dynamical System". Bull. Amer. Math. Soc., 73(1967), 747-817.
- [13] WALTERS, P. "Ergodic Theory Introduction Lectures". Springer Lecture Notes, Vol. 458 (1975).

- [14] WALTERS, P. "Topological conjugacy of Affine Transformation of Tori", Trans. Amer. Math. Soc., Vol. 131, No. 1.
April, 1968.
- [15] WEYL, H. "Mathematische Annalen". Vol. 77 (1916), 313-352.
- [16] ZARISKI, O. and SAMUEL, P. "Commutative algebra", D. Van Nostran, 1958-60.